BIRKHÄUSER

Applied and Numerical Harmonic Analysis

Recent Developments in Real and Harmonic Analysis

In Honor of Carlos Segovia

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Boston • Basel • Berlin

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ISBN 978-0-8176-4531-1 e-ISBN 978-0-8176–4588-5
DOI 10.1007/978-0-8176-4588-5
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2009936079

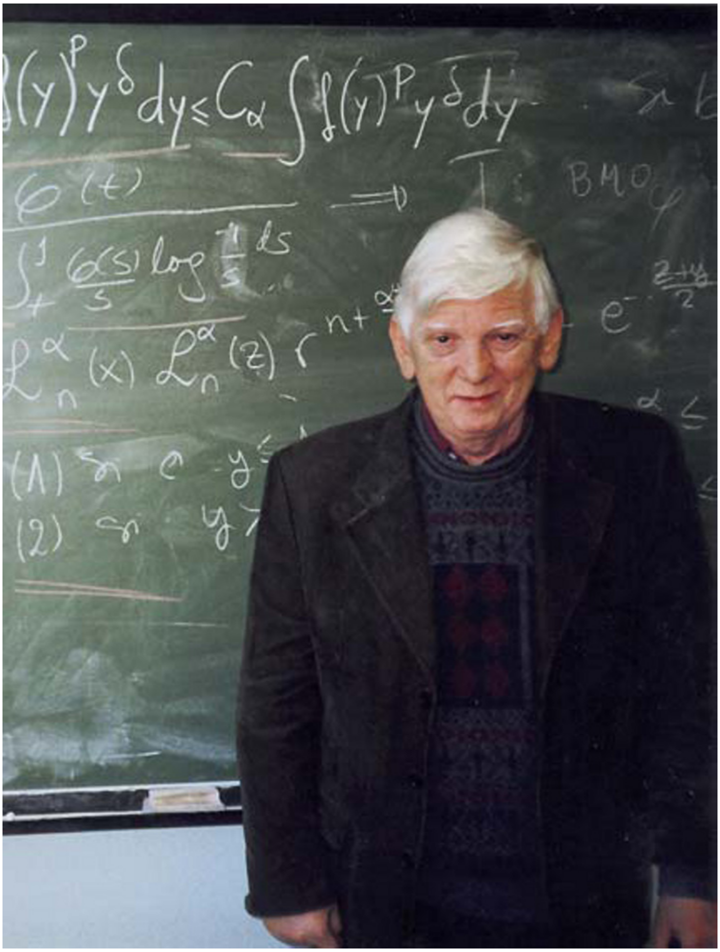
Mathematics Subject Classification (2000): 42xx

Cover design: Alex Gerasev

Printed on acid-free paper

Birkhäuser Boston is part of Springer Science+Business Media (www.birkhauser.com)

A nuestro querido Carlos Segovia



Segovia during a visit to the Math Department at the Universidad Autónoma de Madrid in December 2004.

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

John J. Benedetto
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Foreword

On April 3, 2007, Professor Carlos Segovia passed away in Buenos Aires, Argentina.

He graduated with a degree in Mathematics from the University of Buenos Aires in 1961. In 1967 he received his Ph.D. from the University of Chicago under the direction of Alberto P. Calderón. After two years in a postdoctoral position at Princeton University, he decided to return to Buenos Aires, Argentina. It is there that he spent his professional and political career (except for the period from 1975 to 1979, which he spent at the Universidade Estadual de Campinas, São Paulo, Brazil) at both the University of Buenos Aires and the Instituto Argentino de Matemática (IAM). He held important positions at both institutions: he was president of the University of Buenos Aires during 1982 and the director of the IAM from 1991 to 1998. In 1988 he was chosen to become a member of the Natural and Exact Sciences Academy of Argentina. In 1996 the Third World Academy of Sciences in Trieste (Italy) awarded him the Award in Mathematics.

Professor Segovia's mathematical work falls into the framework of harmonic analysis of the University of Chicago School of the 1960s. He started following in the footsteps of A.P. Calderón and A. Zygmund and continuously extended the subject during the last century. In his Ph.D. thesis, he characterized Hardy spaces of harmonic functions in dimension n by means of square functions. This contribution to the theory of Hardy spaces was very deep and is reflected in the main references of the subject. But without a doubt, his main contribution to mathematics is the sequence of articles, written in collaboration with his colleague and friend Roberto Macías, about spaces of homogeneous type. A space of homogeneous type is a quasi-metric space (without any additional structure) on which one can define a measure which possesses a certain doubling property; precisely, the measure of each ball of radius 2R is controlled by the measure of the ball of radius R (except for a constant that depends on the space). Macías and Segovia developed a satisfactory theory of Hardy spaces and Lipschitz functions in this context. Their ideas not only answered some of the relevant questions about Hardy spaces of that time but also inspired new branches and work in areas of harmonic analysis. “Macías–Segovia” is even today a must-know reference for papers on the subject.

Carlos Segovia was born in Valencia on December 7, 1937. His father was a physician in the army of the Spanish Republic, and his mother’s father came from Ferrol, Galicia. At the end of the Spanish civil war, the family had to leave Spain, and they established themselves in Argentina. Carlos Segovia spent his childhood and youth in Buenos Aires. His Spanish ancestry made the nickname “gallego” unavoidable in Argentina, but his strong “porteño” accent got him called “the Argentinian” in many places in Spain.

In 1988 he was appointed visiting professor by the Universidad Autónoma de Madrid, and from then on he very frequently visited the mathematical department of that university. On his many trips to Spain, he also visited several other universities, with Málaga and Zaragoza being his special favorites. He also was supported by the program for sabbatical leaves of the Education and Science Ministry of Spain and by research grants from the Spanish Ministry of External Affairs. In 1992 he received the title “Catedrático de Universidad” with a contract for five years.

Segovia’s health was relatively weak. In 1999 he suffered a stroke with complications that had him suffering for almost a year, spending several months in the hospital. At some moments he seemed close to death, but each time he survived with very difficult and painful recoveries. The consequences of that period (almost total paralysis of his right side and severe diabetes) did not interfere with his continuing to be the perfect collaborator and a scholarly person of good manners with an exquisite sense of humor. Moreover, he continued his research in mathematics with even stronger vigor; in fact, during his last years he maintained a twofold research agenda. With a group of students in Argentina, he developed a whole theory of “lateral Hardy spaces.” With collaborators from Argentina and Spain, traveling in spite of his physical difficulties at least once a year to Madrid from Buenos Aires, he worked on operators associated with generalized Laplacians. The effort involved in this recent work is, on the one hand, rewarded by the fact that many of his contributions in both lines of research have been published, or will be in the near future, in journals of high level. On the other hand, this effort increases our sadness that he is no longer with us.

Buenos Aires and Madrid
June 2009

Carlos Cabrelli
José Luis Torrea

Preface

From December 12 to 15, 2005, a number of harmonic analysts from all over the world gathered in Buenos Aires, Argentina, for a conference organized to honor Carlos Segovia Fernández on the occasion of his sixty-eighth birthday for "his mathematical contributions and services to the development of mathematics in Argentina."

The conference took place at the Instituto Argentino de Matemática (IAM). The members of the advisory committee were Luis Caffarelli, Cristian Gutiérrez, Carlos Kenig, Roberto Macías, José Luis Torrea, and Richard Wheeden. The local organizing committee members were Gustavo Corach (Chair), Carlos Cabrelli, Eleonor Harboure, Alejandra Maestripieri, and Beatriz Viviani.

Unfortunately, Segovia was not able to attend due to health problems. The conference atmosphere was full of emotion, and many fond memories of Carlos were recalled by the participants.

It was at this meeting that the idea crystallized of writing a mathematical tour of ideas arising around Segovia's work. Unfortunately, one year after the conference, Carlos passed away and could not see this book finished.

The book starts with a chronological description of his mathematical life, entitled "Carlos Segovia Fernández." This comprehensive presentation of his original ideas, and even their evolution, may be a source of inspiration for many mathematicians working in a huge area in the fields of harmonic analysis, functional analysis, and partial differential equations (PDEs). Apart from this contribution, the reader will find in the book two different types of chapters: a group of surveys dealing with Carlos' favorite topics and a group of PDE works written by authors close to him and whose careers were influenced in some way by him.

In the first group of chapters, we find the contribution by Hugo Aimar related to spaces of homogeneous type. Roberto Macías and Carlos Segovia showed that it is always possible to find an equivalent quasi-distance on a given space of homogeneous type whose balls are spaces of homogeneous type. Aimar uses this construction to show a stronger version of the uniform regularity of the balls. Two recurrent topics in the work of Carlos Segovia were commutators and vector-valued analysis, and this pair of topics is the subject of the chapter by Oscar Blasco. He presents part of the work by Segovia related to commutators, and he extends it to a general class of Calderón–Zygmund operators. The words "Hardy, Lipschitz, and BMO" spaces were again recurrent in the work of Segovia. An analysis of the behaviour of the product of a function in some Hardy space with a function in the dual (Lipschitz space) is made in the chapter by Aline Bonami and Justin Feuto. In the last fifteen years Segovia was very interested in applying some of his former ideas in Euclidean harmonic analysis to different Laplacians. He made some contributions to the subject, as can be observed in the publications list included in the present book. Along this line of thought is the chapter by Liliana Forzani, Eleonor Harboure, and Roberto Scotto. They review some aspects of this "harmonic analysis" related to the case of Hermite functions and polynomials. The last Ph.D. students of Segovia were introduced by him to the world of "one-sided" operators, with special attention to weighted inequalities. Francisco Martín-Reyes, Pedro Ortega and Alberto de la Torre survey this subject in their chapter. As the authors say, they try to produce a more or less complete account of the main results and applications of the theory of weights for one-sided operators.

In the second group of chapters, the reader will find the chapter by Luis Caffarelli and Aram Karakhanyan dealing with solutions to the porous media equation in one space dimension. Topics such as travelling fronts, separation of variables, and fundamental solutions are considered. The chapter by Sagun Chanillo and Juan Manfredi considers the problem of the global bound, in the space L^2, of the Hessian of the solution of a certain second-order differential operator in a strictly pseudo-convex pseudo-Hermitian manifold. In the classical case, this global bound can be seen as a "Cordes perturbation method" of the boundedness of the iteration of the Riesz transforms. Well-posedness theory of the initial-value problem for the Kadomtsev–Petviashvili equations is treated in the chapter by Carlos Kenig; a connection with the Korteweg–de Vries equation is also discussed. A survey of recent results on the solutions and applications of the Monge–Ampère equation is written by Cristian Gutiérrez.

We thank all the contributors of this volume for their willingness to collaborate in this tribute to Carlos Segovia and his work.

We are grateful to John Benedetto for inviting us to include our book in his prestigious series *Applied and Numerical Harmonic Analysis*, to Ursula Molter and Michael Shub for their proofreading and helpful comments, and to Tom Grasso and Regina Gorenshteyn from Birkhäuser for their editorial help.

Buenos Aires and Madrid
June 2009

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Publications of Carlos Segovia

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It is an entirely vain endeavor to try to describe in a few pages the mathematical life of Carlos Segovia. The variety and richness of his deep results and proofs would need a whole book in order to put them into their proper context and to see what has been their ulterior influence. However, space limitation has the advantage that this chapter will probably be read by more people than if the exposition was as exhaustive as it deserves. This chapter has been written keeping in mind the idea of reaching more people than just the specialists. It can be thought of as a painting in which only a reduced number of strokes have been made, enough to give the viewer a rough idea of the completed work, but giving him the freedom to choose to finish those parts in which he is more interested.

The order of the exposition is supposed to be chronological when possible. It intends to give a quick presentation of the keystone results of Carlos Segovia; the proofs are omitted and we refer the interested reader to the original papers. The presentation starts with the fundamental period of his Ph.D. studies, which has a crucial influence on the remainder of his work, and it ends with a small exposition of the ideas which Carlos Segovia was dealing with in the very last period of his life. The chapter is divided into the following sections: Square functions, Spaces of homogeneous type, Weighted inequalities, One-sided operators, Vector-valued Fourier analysis, and Harmonic analysis associated with generalized Laplacians.

Carlos Segovia understood the work of a mathematician as a way to relish his lifetime surrounded by friends. In his career he had more than twenty co-authors; all of them had spent long evenings of hard work in which Carlos was struggling very hard with a stubborn result. He is standing in front of the blackboard with a cigarette in one hand and the chalk in the other. Clearly, this presentation is influenced by the author's personal experience and probably it would be different for any of the others co-authors of Professor Segovia. In spite of the space limitations, our intention is to review some of Segovia's cooperation with each of his co-authors.

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_1,

We mention that even before engaging himself in doctoral studies Segovia produced original mathematical results. His first co-authors were A. Benedek and R. Panzone, see [4] and [44].

1 Square functions

Let us consider the area function of Lusin given by

$$S_a(F)(x) = (\int_{\Gamma_a(x)} |F'(u+is)|^2\, du\, ds)^{1/2},$$

where $\Gamma_a(x) = \{(u,s) : |u-x| < as\}$ is the cone in $\mathcal{R}^2_+$ with vertex in the point $(x,0)$ and opening a. In 1965, A. P. Calderón, [6], proved the following characterization of the spaces H^p of analytic functions.

Theorem 1.1 *Let $F(t+is)$ be analytic in $s>0$ and belong to H^p, $0<p<\infty$, that is, $\|F\|_{H^p} = \sup_{s>0}(\int_{-\infty}^{\infty} |F(t+is)|^p\, dt)^{1/p} < \infty$. Then, there exist two positive constants c_1 and c_2, depending on a, and p only, such that*

$$c_1\|F_0\|_{L^p(\mathcal{R})} \le \|S_a(F)\|_{L^p(\mathcal{R})} \le c_2\|F_0\|_{L^p(\mathcal{R})},$$

where $F_0(t) = \lim_{s\to 0} F(t+is)$.

The function S_a is built in three steps: we begin by squaring a quantity, afterwards we integrate, and finally we take the square root. These three steps: square, integration, and square root are the essentials features of a huge family of functions and operators that are known by the generic name of "square functions." These square functions appeared in the 1920s and since then they have been the objects of great interest by people working in probability and harmonic analysis. In the work involving these functions one can find techniques of real and complex variables, functional analysis, probability, and Fourier series. It could be said that they produce a kind of fascination. As an example of their reputation among mathematicians, we quote two instances:

1. B. Bollobás speaking about a result related to some square function, see [5]: "The first result of this paper was proved in January 1975 in order to engage the interest of Professor J.E. Littlewood, who was in hospital at the time"

2. E. M. Stein [78] (monograph about the square functions in the work of A. Zygmund): "A deep concept in mathematics is usually not an idea in its pure form, but rather takes various shapes depending on the uses it is put to. The same is true of square functions. These appear in a variety of forms, and while in spirit are all the same, in actual practice they can be quite different. Thus the metamorphosis of square functions is all important."

Calderón was an expert in the theory of square functions. Theorem 1.1 is a clear example of its use; moreover, he also proved results intrinsic to the theory. For example, his joint paper, with Agnes Benedek and Rafael Panzone, [3], contains a description of square functions as being linear operators valued in a Hilbert space, and this idea has been fundamental during the last 40 years to handle results in the theory. Alberto Calderón was the adviser of Carlos Segovia, and one of the problems proposed as subjects for his doctoral dissertation, [62], was to extend Theorem 1.1 to systems of conjugated harmonic functions in $\mathcal{R}^n$. As a Ph.D. project it was perfect. It was clear that if Carlos Segovia was able to solve the problem, he would become an expert in complex variables, real variable techniques, and also in the, at that moment, incipient theory of H^p spaces. Segovia published his results in 1969 in *Studia Mathematica,* see [60]. The result was the following.

Theorem 1.2 *Let E_n be the n-dimensional Euclidean space of the n-tuples $x = (x_1, \ldots, x_n)$ of real numbers. Denote by E^+_{n+1} the set $\{(x,t) : x \in E_n, t > 0\}$. Let $F(t,x)$ be a harmonic function defined on E^+_{n+1} with values in a real Hilbert space H and satisfying the following conditions:*

(A) $\sup_{t>0} \int_{E_n} \|F(x,t)\|^p_H \, dx < \infty$, *for some $p > 0$,*
(B) There exists a $\delta_0 < p$ such that $\|F(x,t)\|^\delta$ is subharmonic for $\delta \geq \delta_0$, and
(C) The set of zeros of $F(x,t)$ is a polar set in E^+_{n+1}.

Define the function $S_a(F)$ in E^+_{n+1}

$$S_a(F)(x,t) = \left(\int_{\Gamma_a(x)} \left[\left\| \frac{\partial F(u,s+t)}{\partial t} \right\|^2 + \sum_{i=1}^{n} \left\| \frac{\partial F(u,s+t)}{\partial x_i} \right\|^2 \right] \frac{du\, ds}{s^{n-1}} \right)^{1/2},$$

where $\Gamma_a(x) = \{(u,s) : |x-u| < as\}$. Then there exist two positive constants c_1 and c_2 which depend on a, n, and p only such that

$$c_1 \int_{E_n} \|F(x,t)\|^p \, dx \leq \int_{E_n} \|S_a(F)(x,t)\|^p \, dx \leq c_2 \int_{E_n} \|F(x,t)\|^p \, dx.$$

The theorem is applied to solve the following problem proposed by Calderón. Let $U(t,x)$ be a harmonic function defined on E^+_{n+1} with values in a real Hilbert space $\mathcal{H}$. We say that the gradient $\nabla U(t,x)$ belongs to the class $H^p(\mathcal{H})$, $p > 0$, of Hardy if the following conditions are satisfied.

(i) There exists a constant $K > 0$ such that

$$\int_{E_n} \|\nabla U(t,x)\|^p \, dx \leq K^p$$

for every $t > 0$.

(ii) The limit $\lim_{t \to 0} \nabla U(t,x)$ exists for almost every $x \in E_n$. This limit will be denoted by $\nabla U(0,x)$.

Theorem 1.3 *Let $\nabla U(t,x) \in H^p(\mathcal{H})$, $p > \frac{n-1}{n}$. The area function*

$$S_a(\nabla U)(x) = \Big[\int_{\Gamma_a(x)} \Big\{ \sum_{i=0}^{n} \Big\| \frac{\partial \nabla U(s,u)}{\partial x_i} \Big\|^2 \Big\} \, du\, ds \Big]^{1/2}$$

satisfies the inequalities

$$c_1 \int_{e_n} \|\nabla U(0,x)\|^p \, dx \le \int_{E_n} S_a(\nabla U)^p(x) \le c_2 \int_{e_n} \|\nabla U(0,x)\|^p \, dx,$$

where c_1 and c_2 are two positive constants depending on a, p, and n only.

For a detailed treatment of Theorem 1.3 see Chapter VII in [77].

After completing his Ph.D. thesis, Segovia became interested in the idea of defining derivatives of fractional order. This topic appeared several times during his career. In [75] one can find the following notion of α-fractional derivation. Let $h(x,t) = h(x+it)$ belong to the Hardy space H^p, $p > 0$, that is analytic for $t > 0$ and satisfying $\sup_{t>0} \int_{-\infty}^{\infty} |h(x+it)|^p \, dx \le C$. For $0 < \alpha < 1$, let

$$h^{(\alpha)}(x,t) = \int_0^\infty \frac{\partial h}{\partial t}(x, t+s) s^{-\alpha} \, ds. \tag{1}$$

A motivation for the above definition could be the following. Let us write a harmonic function $u(x,t)$ as a convolution of a function $f(x)$ with the Poisson kernel, $u(x,t) = P_t * f(x)$. Using the spectral theorem, this convolution can be expressed as $u(x,t) = e^{-t\sqrt{-\Delta}} f(x)$ where Δ is the Laplacian in one dimension. Hence, at least formally, we have

$$\begin{aligned} \int \frac{\partial}{\partial t} u(x, t+s) s^{-\alpha} \, ds &= \int \frac{\partial}{\partial t} e^{-(s+t)\sqrt{-\Delta}} f(x) s^{-\alpha} \, ds \\ &= -\int \sqrt{-\Delta}\, e^{-t\sqrt{-\Delta}} e^{-s\sqrt{-\Delta}} f(x) s^{-\alpha} \, ds \\ &= -C\, e^{-t\sqrt{-\Delta}} (\sqrt{-\Delta}\,)^\alpha f(x) \\ &= -C\, (\sqrt{-\Delta}\,)^\alpha e^{-t\sqrt{-\Delta}} f(x). \end{aligned}$$

In collaboration with R. Wheeden, Carlos Segovia published a series of papers, [73, 74, 75], in which the square functions and the fractional integrals are used in order to obtain several deep results. We shall now briefly describe some of them.

Given a function $h(x,t)$, $x \in \mathcal{R}$, $t > 0$, define the area function of fractional order γ, $0 < \gamma < 1$ as

$$S^{(\gamma)}(h)(x) = \left(\int_{\Gamma(x)} |s^\gamma \, h^{(\gamma)}(u,s)|^2 \frac{du\, ds}{s^2} \right)^{1/2},$$

where $h^{(\gamma)}$ denotes the "γ-derivative" in (1) and as usual $\Gamma(x)$ denotes the cone. The following theorem is proved in [73].

Theorem 1.4 *Let $h(x,y), y>0$ belong to H^p for some $p>0$ and let $h(x) = \lim_{y\to 0} h(x,y)$. There exist positive constants c_1 and c_2 independent of h such that*

$$c_1\|h\|_p \le \|S^{(\gamma)}(h)\|_p \le c_2\|h\|_p.$$

This result is used in order to give a characterization of potential spaces L^p_α. It is well known that these spaces coincide with Sobolev spaces whenever $\alpha \in \mathcal{N}$. By L^p_α, $1\le p\le\infty, \alpha>0$ we denote the class of functions which are Bessel potentials of order α of L^p functions, and let $\|\cdot\|_{p,\alpha}$ denote the norm in L^p_α, see [77]. The next result can also be found in [73].

Theorem 1.5 *Let k be a positive integer and $f\in L^p, 1<p<\infty$. Let $f(x,t)$, $t>0$, denote the Poisson integral of f and for $0<\alpha<k$ set*

$$S_{(\alpha,k)}(f)(x) = \left(\int_{\Gamma(x)} \left|s^{(k-\alpha)}\,\frac{\partial^k f(u,s)}{\partial s^k}\right|^2 \frac{du\,ds}{s^{n+1}}\right)^{1/2}.$$

Then if $f\in L^p_\alpha$,

$$c_1\|f\|_{p,\alpha} \le \|S_{(\alpha,k)}(f)\|_p + \|f\|_p \le c_2\|f\|_{p,\alpha}.$$

Moreover, if f is any L^p function, $1<p<\infty$, and $S_{(\alpha,k)}(f)\in L^p$, then $f\in L^p_\alpha$.

It must be noticed here that in proving Theorem 1.4 different "square functions" were considered for a given function $h(x,t)$, $x\in\mathcal{R}^n$, $t>0$, and a fractional derivative γ. Namely,

$$g^{(\gamma)}(h)(x) = \Big(\int_0^\infty |t^\gamma h^{(\gamma)}(x,t)|^2 \frac{dt}{t}\Big)^{1/2}$$

$$g^{*,\gamma}_\lambda(u)(x,t) = \Big(\int\int_{t>0} \frac{t^{\lambda n}}{(|x-y|+t)^{\lambda n}} |t^\gamma h^{(\gamma)}(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2}.$$

The corresponding versions for $\gamma=1$, that is, the case of the standard derivative, were also treated.

The same authors, using complex variable methods and some fractional area integrals, proved results about commutators. We shall give a short outline of them.

If

$$Hf(x) = p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy$$

denotes the Hilbert transform of f and $Af(x)=a(x)f(x)$ denotes the pointwise multiplication, we form the commutator

$$[A,H]f(x) = (AH-HA)(f)(x) = p.v. \int_{-\infty}^{\infty} \frac{a(x)-a(y)}{x-y} f(y)\,dy. \qquad (2)$$

This operator appeared in some work by Calderón, see [6]. The following result can be found in [75].

Theorem 1.6 *Let* $f \in L^p$, $1 < p < \infty$, *and* $a \in L^\infty_\alpha$, $0 < \alpha < 1$. *Then the commutator* $(AH - HA)f \in L^p_\alpha$. *Moreover,*

$$\|(AH - HA)f\|_{p,\alpha} \leq C\|f\|_p\|a\|_{\infty,\alpha},$$

where C *is independent of* α *and* f.

2 Spaces of homogeneous type

A. P. Calderón and A. Zygmund developed in the 1950s a theory of singular integrals, that is, operators of the form

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\{|x-y|>\varepsilon\}} k(x-y)f(y)dy + cf(x),$$

where f is a function on $\mathcal{R}^n$, $k(x)$ is a homogeneous function of degree $-n$, satisfying $\int_{\{x:|x|=1\}} k(x)\,dx = 0$, and c is a constant. Professor Calderón in his speech of acceptance as an Academician of the "Academia de Ciencias Exactas, Físicas y Naturales de Argentina" said that these singular integrals can be considered as hybrid objects between integral operators and differential operators. His whole thought was presented as follows.

"Los operadores integrales singulares pueden ser concebidos como entes híbridos o intermedios entre los diferenciales y los integrales. Son operadores integrales lineales en que las expresiones a integrar dan lugar a integrales divergentes, es decir, sin sentido como integrales ordinarias y a las cuales es necesario definir como límite de integrales comunes. Más especificamente, las expresiones a integrar tienen el orden de magnitud de la inversa de la distancia a un punto elevada a la dimensión del espacio o número de variables de las funciones sobre las que se opera, lo que las situa al borde de la integrabilidad. Esto hace de ellas operadores semilocales, es decir, hay acciones a distancia, pero la acción sobre un punto está dominada por la de puntos arbitrariamente próximos."

The motivation of Calderón and Zygmund was inside the theory of partial differential equations, namely in the fact that every linear (constant coefficients) differential operator can be written as the composition of two operators, one of them being a power of the Laplacian and the other one a singular integral operator. Calderón and Zygmund proved that these operators were continuous in the Lebesgue spaces $L^p(\mathcal{R}^n)$, $1 < p < \infty$, see [7], [8]. Some "a priori" estimates are easily obtained from this result. For differential operators with smooth enough coefficients an error term appears, see [79].

This fundamental work immediately stimulated the interest of the mathematical community, and a lot of new results and applications were obtained. Carlos Segovia participated actively in two different aspects of this new current:

(1) Extension of the theory to spaces in which there are privileged directions and, even more, extension to spaces without a vector space structure. Later on, these two kind of extensions were called spaces of mixed homogeneity and spaces of homogeneous type; some papers by Segovia in this direction are [74] [76],[2], and [32].
(2) The spaces in which the distances considered have to be modified in the sense that the measures were allowed to have a density with respect to the Lebesgue measure. That is, $L^p(\mathcal{R}^n, \omega(x)dx)$-norms of the functions are defined with respect to some "weight" ω. This weighted theory began essentially in a paper by B. Muckenhoupt, [41], and later on had a development parallel to the theory of singular integrals.

In this section we shall comment on part of the work of Professor Segovia related to item (1). The idea of developing a huge part of harmonic analysis in spaces without vector space structure goes back to the end of the 1970s. There were several publications addressing that problem, but probably the monograph [13] was the most relevant. In this theory there is a lack of the concept of derivative, therefore some extensions of essential parts of the classical harmonic analysis (like, for example, H^p spaces) were considered a rather difficult obstacle. We shall present a brief account of the results obtained in this direction by C. Segovia jointly with R. Macías.

Let $d(x, y)$ (quasi-distance) be a non-negative function defined on $X \times X$, such that:

(i) for every $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,
(ii) for every $x, y \in X$, $d(x, y) = d(y, x)$, and
(iii) there exists a finite constant K such that for every x, y and z in X

$$d(x, y) \leq K \Big(d(x, y) + d(y, z) \Big).$$

The balls $B(x, r) = \{y : d(x, y) < r\} \quad r > 0$ form a basis of neighborhoods of x for the topology induced by d. Two quasi-distances d and d' are equivalent if there exist two positive constants c_1 and c_2 such that $c_1 d(x, y) \leq d'(x, y) \leq c_2 d(x, y)$, for every $x, y \in X$. The topologies defined by equivalent quasi-distances coincide.

Let X be a set endowed with a quasi-distance d. Let μ be a non-negative measure defined on a σ-algebra of subsets of X which contains the d-open subsets and the balls $B(x, r)$. Assume that there exist two finite constants, $a > 1$ and A, such that

$$0 < \mu(B(x, ar)) \leq A \cdot \mu(B(x, r)) < \infty$$

holds for every $x \in X$ and $r > 0$. A set X with a quasi-distance $d(x, y)$ and a measure μ satisfying the conditions above will be called a space of homogeneous type and denoted by (X, d, μ). We shall say that the space is normal if there exist two positive and finite constants c_1 and c_2, satisfying

$$c_1\, r \le \mu(B(x,r)) \le c_2\, r \tag{3}$$

for every $x \in X$ and every r, $\mu(\{x\}) < r < \mu(\{X\})$.

The following theorem, which can be found in [33] (Theorems 2 and 3), analyses the possibility of working with smooth quasi-distances having a close relation with the measure.

Theorem 2.1 *Let (X, d, μ) be a space of homogeneous type. Then*

(1) There exists a quasi-distance d' on X, such that
(i) d' is equivalent to d, and
(ii) there exist a finite constant C and a number $0 < \alpha < 1$, such that

$$\Big| d'(x,y) - d'(y,z) \Big| \le Cr^{1-\alpha}(d'(x,y))^{\alpha}$$

holds whenever $d'(x,z)$ and $d'(y,z)$ are both smaller than r.

(2) The function $\delta(x,y)$ defined as

$$\delta(x,y) = \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\},$$

if $x \ne y$, and $\delta(x,y) = 0$, if $x = y$, is a quasi-distance on X. Moreover, (X, δ, μ) is a normal space and the topologies induced on X by d and δ coincide.

As we said before, the absence of a vector structure produces a lack of derivatives. The role of smooth functions is taken by some classes of Lipschitz functions. Macías and Segovia considered two different families of these functions. Given a function ϕ, integrable on bounded subsets, and a ball B, consider the number

$$m_B(\phi) = \frac{1}{\mu(B)} \int_B \phi(x)\, d\mu(x).$$

Let $1 \le q < \infty$ and $0 < \beta < \infty$. We say $\phi \in \mathrm{Lip}(\beta, q)$, $1 \le q < \infty$, if there exists C such that

$$\left(\frac{1}{\mu(B)} \int_B |\phi(x) - m_B(\phi)|^q d\mu(x) \right)^{1/q} \le C\mu(B)^{\beta}$$

holds for every ball B. In the case $q = \infty$ we say $\phi \in \mathrm{Lip}(\beta, \infty)$ if

$$\text{ess sup}_{x\in B} |\phi(x) - m_B(\phi)| \le C\mu(B)^{\beta}$$

holds for every ball B. The least constant C satisfying the conditions above shall be denoted by $\|\phi\|_{\beta,q}$.

A "pointwise" Lipschitz condition is also considered. We say that a function ϕ belongs to $\mathrm{Lip}(\beta)$, $0 < \beta < \infty$, if there exists a constant C satisfying

$$|\phi(x) - \phi(y)| \le Cd(x,y)^{\beta}$$

for every x, $y \in X$. Again, $\|\phi\|_\beta$ stand for the least constant C satisfying the condition above.

The following theorem establishes the key relation between both Lipschitz concepts, through the notion of normal space; see (3).

Theorem 2.2 *Let (X, d, μ) be a space of homogeneous type. Then, given $0 < \beta < \infty$, there exists a quasi-distance $\delta(x, y)$ on X such that (X, δ, μ) is a normal space and satisfies the following property. For any $1 \leq q \leq \infty$ a function ϕ belongs to* $\mathrm{Lip}(\beta, q)$ *if and only if ϕ is equal almost everywhere to a function ψ which belongs to* $\mathrm{Lip})(\beta)$ *on (X, δ, μ). Moreover, the norms $\|\phi\|_{\beta,q}$ and $\|\psi\|_\beta$ are equivalent.*

The final purpose of Macías and Segovia was to develop a satisfactory theory of Hardy spaces H^p. In order to have success, a substitute of C^∞ functions with compact support and of distributions was needed. Let us see how this was done in [34].

We are given a point x_0 in X and a positive integer n. We denote by E_n^α the set of all functions which belong to $\mathrm{Lip}(\beta)$ for all $0 < \beta < \alpha$ and with support contained in $B(x_0, n)$. Consider the topology Z_n^α defined by the family of norms $\{\, \| \cdot \|_{\mathrm{Lip}(\beta)} : 0 < \beta < \alpha \,\}$ and $\| \cdot \|_\infty$. The space (E_n^α, Z_n^α) is a Fréchet space and the topology Z_{n+1}^α restricted to E_n^α coincides with Z_n^α. Denote by E^α the strict inductive limit of the sequence $\{E_n^\alpha\}_{n=1}^\infty$. The definition of E^α does not depend on the point x_0 chosen in the definition of the spaces E_n^α. We shall say that a linear functional is a distribution on E^α if it is continuous.

For γ, $0 < \gamma < \alpha$, and $x \in X$, we shall say that a function ψ belonging to E^α is in $T_\gamma(x)$ if there exists r such that the support of ψ is contained in $B(x, r)$ and

$$\|\psi\|_\infty \leq \frac{1}{r}, \quad \|\psi\|_{Lip(\gamma)} \leq \frac{1}{r^{1+\gamma}}.$$

Given a distribution f on E^α and $0 < \gamma < \alpha$, the γ-maximal function $f_\gamma^*(x)$ is defined as

$$f_\gamma^*(x) = \sup\{|\langle f, \psi\rangle| : \psi \in T_\gamma(x)\}.$$

Finally, the Hardy-type space H^p is the set of distributions on E^α such that for some γ, $0 < \gamma < \alpha$, and some p, $(1+\gamma)^{-1} < p \leq 1$, its γ-maximal function $f_\gamma^*(x)$ belongs to $L^p(X, d\mu)$. For these distributions a decomposition in a series of "atoms" is found. Some approximation results by Lipschitz functions are also obtained.

Later on, Carlos Segovia made even more contributions to the theory of spaces of homogeneous type. One of particular interest is the somehow generalization of the fact that the composition of the inverse of the Laplacian with a fractional integral of certain order is a singular integral. In [19] "fractional derivatives" D_α and "fractional integrals" I_α are defined in the context of spaces of homogeneous type as follows:

$$D_\alpha f(x) = \int_X \frac{f(y) - f(x)}{\delta(x, y)^{1+\alpha}} d\mu(y) \quad , \quad I_\alpha f(x) = \int_X \frac{f(y)}{\delta(x, y)^{1-\alpha}} d\mu(y).$$

It is proved that the composition operator $D_\alpha \circ I_\alpha$ is bounded in L^2. Also, the kernel satisfies the correct condition in order to apply the theory of Calderón and Zygmund in this setting. Due to the nonexistence of a Plancherel-type theorem, the proof of the boundedness in L^2 needs to resort to a $T1$ criterion. Professor Segovia had previously been engaged in the study of this criterion in the context of spaces of homogeneous type in [36].

3 Weighted inequalities

Carlos Segovia became interested in weighted inequalities from the very beginnings of the theory. His interest in the subject was of a rather general nature, and he got results in varied directions on the theme. Foremost, in collaboration with R. Wheeden [76] and N. Aguilera [2], he published some results comparing L^p-weighted norms of different square functions either among themselves or with some maximal functions. This work, in the words of the authors, can be seen as *"a step in finding weighted versions of Littlewood–Paley theorems."* In [76] it is proved that if a weight $\omega(\theta)$ in the torus satisfies

$$\int_0^{2\pi} |\widetilde{f}(\theta)|^p \omega(\theta) d\theta \le C \int_0^{2\pi} |f(\theta)|^p \omega(\theta) d\theta, \quad 1 < p < \infty, \tag{4}$$

where $\widetilde{f}$ is the conjugate function, i.e.,

$$\tilde{f}(\theta) = p.v. \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(\theta - \phi)}{2\tan(\phi/2)} d\phi, \tag{5}$$

then ω satisfies

$$\int_0^{2\pi} |S(f)(\theta)|^p \omega(\theta) d\theta \le C \int_0^{2\pi} |f(\theta)|^p \omega(\theta) d\theta.$$

$S(f)(\theta)$ is the Lusin area function defined in the disc as

$$S(f)(\theta) = \Big(\int \int_{\Gamma(\theta)} |(\nabla f)(re^{i\phi})|^2 r dr d\phi \Big)^{1/2}, \quad 1 < p < \infty, \tag{6}$$

where $\Gamma(\theta) = \Gamma(\theta, \delta), 0 < \delta < 1$, is the open canonical region bounded by the two tangents from $e^{i\theta}$ to $|z| = \delta$ and the more distant arc of $|z| = \delta$ between the points of contact. The class of weights that satisfy (4) had been characterized in [31] by an intrinsic condition called A_p. Namely, ω satisfies

$$\Big(\frac{1}{|I|} \int_i \omega(\theta) d\theta \Big) \Big(\frac{1}{|I|} \int_I \omega(\theta)^{-1/(p-1)} d\theta \Big)^{p-1} \le c, \tag{7}$$

with c independent of I, for every interval I which has length less than or equal to 2π and center in $(0, 2\pi)$. Hence, the result in [76] can be stated as "if a weight ω satisfies condition A_p, then it satisfies inequality (5)."

Also, in the beginnings of the theory and in collaboration with R. Macías, he got results about parabolic fractional integrals of distributions in [32].

He came back to weight theory in 1985 with a series of three papers, [23], [24] and [25], concerning two different aspects of the theory that had just been raised by Rubio de Francia, namely

(1) The "two weights problem" and
(2) "Extrapolation" of A_p-weighted inequalities.

The"two weights problem" can be formulated as follows. Given an operator T bounded from L^p into L^q for some p, q, give necessary and sufficient conditions on a weight ω (resp. v) for the existence of a nontrivial weight v (resp. ω) such that

$$\Big(\int |Tf(x)|^q v(x)\,dx\Big)^{1/q} \leq C\Big(\int |f(x)|^p \omega(x)\,dx\Big)^{1/p}$$

holds. This topic is considered in [23] and [24] and the conditions for the weights are found in the case of $T = I_\alpha$, the fractional integral in $\mathcal{R}^n$ defined by $I_\alpha f(x) = \int_{\mathcal{R}^n} f(y)|x-y|^{\alpha-n}\,dy$, $0 < \alpha < n$, and p, q satisfying $1/q \geq 1/p - \alpha/n$.

By "extrapolation" of A_p-weighted inequalities, see [55] and [56], we mean the following. If for some $1 \leq p_0 < \infty$ an operator preserves $L^{p_0}(\omega)$ for every $\omega \in A_{p_0}$, then, for every $1 < p < \infty$, it preserves $L^p(\omega)$ for every $\omega \in A_p$. By A_p we denote the class of weights defined in $\mathcal{R}^n$ analogously as in the torus by (7). Segovia in [25] extends this result in two directions: on one hand he considers (L^p, L^q)-boundedness, and on the other hand he considers the more useful limiting case $p = \infty$. Here are the results.

Let T be a sublinear operator defined in $C_0^\infty(\mathcal{R}^n)$. If $\|(Tf)v\|_q \leq C(v)\|fv\|_p$ holds for some pair (p_0, q_0), $1 < p_0 \leq q_0 < \infty$, and every weight $v \in A(p_0, q_0)$, then it also holds for any other pair (p, q), $1 < p \leq q < \infty$, satisfying $1/p - 1/q = 1/p_0 - 1/q_0$, and any weight $v \in A(p, q)$. For the limiting case $q_0 = \infty$, $1 < p_0 \leq \infty$, the result is still valid in the following form.

Theorem 3.1 *Let T be a sublinear operator defined in $C_o^\infty(\mathcal{R}^n)$. If*

$$\|v\chi_Q\|_\infty \left(\frac{1}{|Q|}\int_Q |Tf(x) - m_q(Tf)|\,dx\right) \leq C(v)\|fv\|_\infty$$

holds for every cube Q and any weight v with $v^{-1} \in A_1$, then for every $w \in A_p$, $1 < p < \infty$ we have

$$\int |Tf|^p w\,dx \leq C(w)\int |f|^p w\,dx.$$

Later on he made further contributions to the theory of weighted inequalities as we shall see in this chapter.

4 One-sided operators

In 1986, see [58], E. Sawyer characterised the weights ω on $\mathcal{R}$ for which the "one-sided" maximal operator,

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\, dy,$$

maps either $L^p(\mathcal{R}, \omega)$, $1 < p < \infty$, into itself or $L^1(\mathcal{R}, \omega)$ into weak $L^1(\mathcal{R}, \omega)$. This maximal operator intrinsically involves the notions of future and past. Obviously, the operator M^+ is controlled pointwise by the centered maximal operator defined as

$$Mf(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x+h} |f(y)|\, dy.$$

The novelty in the work of Sawyer was to see that the weights ω for which M^+ is bounded in $L^p(\mathcal{R}, \omega)$ were of a class different from the class of weights corresponding to M. We want to mention here that the study of the operator M^+ has as a by-product some nontrivial results in ergodic theory. This pioneer work of Sawyer opened the door to pose diverse questions about operators of the type

$$T(f)(x) = T(\chi_{[x,\infty)} f)(x). \tag{8}$$

Observe that $M^+(f)(x) = M^+(\chi_{[x,\infty)} f)(x)$. In particular, in the L^1 level, a kind of H^1 question could be: *find the class of functions* $f \in L^1(\omega)$ *such that* $Tf \in L^1(\omega)$, where T is an operator as in (8). Segovia analyzed this question in connection with the theory of one-sided H^p spaces and the operator T defined as a convolution with the kernel $K(x) = \dfrac{e^{i|x|^{-1}}}{|x|}\chi_{(\infty,0]}(x)$.

Given a distribution F on the real line and a positive integer γ, the maximal function $F^*_{+,\gamma}$ is defined as $F^*_{+,\gamma}(x) = \sup |< F, \psi >|$, where the sup is taken over test functions ψ whose support is contained in a compact interval I_ψ, with $x \leq \inf I_\psi$, and such that $\|D^{\gamma+1}\psi\|_\infty \leq 1/\|I_\psi\|^{\gamma+1}$.

Given p and q satisfying either $0 < p \leq 1$, $(\gamma+1)p \geq q > 1$ or $(\gamma+1)p > 1$ if $q = 1$ and $w \in A^+_q$, $H^p_{+,\gamma}(w)$ is defined as the class of distributions F such that

$$\int_R (F^*_{+,\gamma}(x))^p w(x)\, dx < \infty.$$

Some atomic decompositions were found jointly with some one-sided versions of area function and harmonic majorization. In the last 15 years, Carlos Segovia in collaboration with L. de Rosa, S. Ombrosi, and R. Testoni was developing a kind of "one-sided" harmonic analysis on the real line. Their results can be found in several publications, [52], [49], [50], [48], [54] ,[53], [43], [65], [66], and [42]. These papers are, in general, concise, but nonetheless all of them contain the results needed to follow, in a relatively smooth way, the computations. Segovia and his collaborators are able to develop a theory of one-sided weighted H^p spaces. In developing this theory, they must introduce concepts of "lateral" square functions and "lateral area" functions. We refer to the survey [39] in this volume for more details in area.

5 Vector-valued Fourier analysis

In February 1988 Carlos Segovia was appointed Visiting Professor in the Universidad Autónoma de Madrid. There he became acquainted with several mathematicians of Rubio de Francia's school, see [57]. Henceforth, he became interested in the theory of vector-valued Fourier analysis. As is known, vector-valued analysis was extensively developed in Spain due to the intensive work of J.L. Rubio de Francia. It has two motivations:

(1) To analyze the classes of Banach spaces for which some classical results that are true for complex functions remain valid for functions whose values are taken in a Banach space.
(2) To apply vector-valued inequalities to obtain inequalities valid for classical operators.

To begin, we shall describe some work by Carlos Segovia inspired by the first motivation, that is, related to the geometry of Banach spaces. Let X be a Köthe–Banach space of equivalence classes, modulo equality almost everywhere, of locally integrable real functions ϕ on a measure space (Ω, μ). Every Köthe function space satisfies the following.

(i) If $|\phi(\omega)| \leq |\varphi(\omega)|$ a.e. ϕ measurable and $\varphi \in X$, then $\phi \in X$ and $\|\phi\| \leq \|\varphi\|$.
(ii) If $\mu(E) < \infty$, then $\chi_E \in X$.

Let X be a Banach lattice and let J be a finite subset of positive rationals. Given a locally integrable X-valued function $f(x), x \in \mathcal{R}^n$, we define

$$\mathcal{M}_J f(x) = \sup_{r \in J} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $|f(x)|$ denotes the absolute value of $f(x)$ in the lattice X. As usual, $B(x,r)$ is a ball of radius r, centered at x, and $|B(x,r)|$ stands for its Lebesgue measure.

The operator $\mathcal{M}f(x) = \sup_J \mathcal{M}_J f(x)$ appears in a natural way for certain Banach lattices, for example, $X = \ell^r, 1 < r < \infty$, and it was known that $\mathcal{M}$ is bounded from $L^p(\mathcal{R}^n, \ell^r)$ into itself. The two weights problem, as defined in Section 3, is considered in [27]. We are trying to find conditions in the weight v and in the lattice X in order to ensure validity of inequalities of the type

$$\sup_J \int_{\mathcal{R}^n} \|\mathcal{M}_J f(x)\|_X^p u(x)\, dx \leq C \int_{\mathcal{R}^n} \|f(x)\|_X^p v(x)\, dx. \tag{9}$$

Inequalities such as (9) had been previously considered in the scalar case ($X = \mathcal{R}$) by different authors, see [55], [16]. The condition found over v was

$$\sup_{R\geq 1} R^{-np'} \int_{|x|\leq R} v(x)^{p'/p}\, dx < \infty, \quad (D_p^*). \tag{10}$$

In the case of the Hilbert transform the problem had been solved also, see [11]. In this case the condition found was

$$\int_{\mathcal{R}^n} (1+|x|)^{-np'} v(x)^{-p'/p}\, dx < \infty, \quad (D_p). \tag{11}$$

In order to solve this problem the following machinery is introduced by Segovia and his collaborators in [27].

Definition 5.1 *Given a finite sequence $\{r_i\}_{i=1}^m$ where $r_i \geq 1$ and $r_i \in \mathcal{Q}_+$, let us denote by B_i the balls $B(0, r_i)$. If $\{\Omega_i\}_{i=1}^m$ is a measurable partition of Ω, we define the function $\varphi(x,\omega)$ as*

$$\varphi(x,\omega) = \sum_{i=1}^m |B_i|^{-1} \chi_{B_i}(x) \chi_{\Omega_i}(\omega).$$

Observe that $\varphi(x,\omega)$ is a step function in ω, for any given x. Moreover, if a belongs to X', then for any given x, $\varphi(x,\omega)a(\omega)$ belongs to X'. We denote this function by $\varphi(x)a$.

Definition 5.2 *Let $v(x)$ be a weight on $\mathcal{R}^n$ and X be a Köthe function space with X' norming; for $1 < p < \infty$, we shall say that $v(x)$ belongs to the class $D(p,X)$ if*

$$\sup_{a\in X',\, \|a\|_{X^*}\leq 1} \int_{\mathcal{R}^n} \|\varphi(x)a\|_{X^*}^{p'} v(x)^{-p'/p}\, dx < \infty.$$

Here is the solution to problem (9).

Theorem 5.3 *Let X be a Köthe function space, with X' norming, and such that inequality (9) holds for $u(x) = v(x) = 1$. Then for every p, $1 < p < \infty$, the following conditions are equivalent.*

(i) $v(x)$ belongs to the class $D(p,X)$.
(ii) There exist a nontrivial weight $u(x)$ and a finite constant c_p, such that inequality (9) holds for every f in $L_X^p(v)$.

We also describe some results that can be found in [27].

Theorem 5.4 *Let X be a Köthe function space with X' norming. Then, we have*

(i) $D(p,X) = D_p^*$ for some p, $1 < p < \infty$, if and only if X is r-convex for some $r > 1$, and

(ii) $D(p,X) = D_p$ *for every* p, $1 < p < \infty$, *if and only if* X *is not* r*-convex for any* $r > 1$.

Special attention is devoted to the classes $D(p,\ell^r)$. The following relation is obtained for $1 < r < s < p$:

$$D_p = D(p,\ell^1) \subsetneq D(p,\ell^r) \subsetneq D(p,\ell^s) \subsetneq D(p,\ell^p) = D(p,\mathcal{R}) = D_p^*.$$

The wide experience of Professor Carlos Segovia in commutators was fundamental in order to develop an exhaustive study of commutators for vector-valued operators. This work can be found in a series of papers, [67], [68], [72], [71], [70] and [29]. The starting point was the following theorem due to Coifman, Rochberg, and Weiss, see [12].

Theorem 5.5 *The commutator* $[A,H]$, *see (2), is bounded from* $L^p(\mathcal{R},dx)$ *into* $L^p(\mathcal{R},dx)$, *for* $1 < p < \infty$, *if and only if* $A \in$ BMO.

We recall the definition of the class BMO. Given a function A and an interval I, we define the mean of A over the interval I as $A_I = \frac{1}{|I|}\int_I A(y)\,dy$ and the "sharp" maximal function as

$$A^\sharp(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |A(x) - A_I|\,dx.$$

We say that $A \in$ BMO if $A^\sharp \in L^\infty(\mathcal{R})$. The norm in the space BMO is defined by $\|A\|_{\mathrm{BMO}} = \|A^\sharp\|_{L^\infty}$.

Segovia observed that a key fact was to prove boundedeness of operators like

$$S_A f(x) = \sup_{x\in I} \frac{1}{|I|}\int_I (A(x) - A(y))f(y)\,dy$$

and

$$\tilde{S}_A f(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |A(x) - A(y)|f(y)\,dy.$$

But he also observed that these operators follow the model $[A,H]$ but from a vector-valued point of view. With this idea in mind, Segovia recovered his "old" commutator problem and, by using ideas of extrapolation of weights, he proved a series of results for a huge class of operators. In order to be concise, we present here just a few examples of his results.

(1) The function $A \in BMO$ if and only if the operators S_A^+ and $\tilde{S}_A$ map L^p into L^p from some p in the range $1 < p < \infty$.

(2) Given a function $A \in BMO$, the following operators are bounded from $L^p(\mathcal{R})$ into itself:

   (i) $f \to \sup \left| \int_{\mathcal{R}} \frac{b(x) - b(y)}{x - y} e^{-iRy} f(y)\,dy \right|$, $1 < p < \infty$, and

(ii) $$f \to \left(\sum_j \left| a(x) S_{I_j} f(x) - S_{I_j}(bf)(x) \right|^2 \right)^{1/2},$$

where $S_I f$ denotes the partial sum operator defined in L^2 as $\widehat{(S_I f)}(\xi) = \chi_I(\xi)\,\widehat{f}(\xi)$. If the family of intervals is the family of dyadic intervals, then the result is true for $1 < p < \infty$. In the case of arbitrary families, the result is valid in the range $2 < p < \infty$.

(3) Let A be a function in $BMO(\mathcal{R}^{n+1})$. Consider the parabolic-type partial differential equation

$$L[u] = \frac{\partial}{\partial t} u - (-1)^{m/2} P(D) u = f,$$

where $P(x)$ is a polynomial of even degree m such that $P(x)$ has negative real part for each real value x. Then, the operator

$$u \to A(x,t) D_x^\rho u(x,t) - D_x^\rho (Au)(x,t)$$

is bounded from $L^p(\mathcal{R}^{n+1})$ into itself, $1 < p < \infty$. Here D_x^ρ is the spatial derivative of order $\rho = (\rho_1, \ldots, \rho_n)$, $|\rho| = \rho_1 + \cdots + \rho_n = m$, of the solution

$$u(x,t) = \int_0^t \int_{\mathcal{R}^n} \Gamma(x-\xi, t-\tau) f(\xi,\tau)\, d\xi\, d\tau,$$

where

$$\Gamma(x,t) = \frac{1}{2\pi} \int_{\mathcal{R}^n} \exp(ix \cdot \xi + tP(\xi))\, d\xi.$$

See the series of papers [69], [67], [68], [72], [71], [70] and [15].

6 Harmonic analysis associated with generalized Laplacians

Consider the differential operator on $\mathcal{R}^n$ given by

$$L = -\frac{1}{2}\Delta + x \cdot \nabla.$$

This operator is symmetric with respect to the (Gaussian) measure $d\gamma(x) = e^{-|x|^2}\, dx$, and in one dimension, the eigenfunctions of the operator L are the family of Hermite polynomials defined in the real line by the formula

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2},\ m = 0, 1, \ldots.$$

Hermite polynomials exist also in dimension n; they are given by tensor products $H_\alpha = \otimes_{i=1}^n H_{\alpha_i}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$. They satisfy $LH\alpha = |\alpha| H_\alpha$. The heat semigroup e^{-tL} is known as the Ornstein-Uhlenbeck semigroup. Some known formulas for the Γ function allow us to define in a natural way the "Poisson" semigroup as

$$e^{-t\sqrt{L}}f = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} \exp(-t^2/4s) e^{-sL} f\, ds, \tag{12}$$

and also "fractional integrals" and L^{-b}, $b > 0$, as

$$L^{-b}f = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL} f t^{b-1}\, dt. \tag{13}$$

Observe that the above formulas applied to H_α, $\alpha = (\alpha_1, \ldots, \alpha_n)$, produce $e^{-tL}H_\alpha = \sqrt{|\alpha|}H_\alpha$ and $L^{-b}H_\alpha = (|\alpha|)^{-b}H_\alpha$.

The operator L can be factored as

$$L = \frac{1}{2} \sum_i \partial_i^* \partial_i,$$

being $\partial_i f = \dfrac{\partial f}{\partial x_i}$ and $\partial_i^* = -\partial_i + 2x_i$, that is the formal adjoint of ∂_i with respect to the Gaussian measure. In these circumstances, following Stein [77] the Riesz transforms can be defined as

$$R_i = \partial_i \circ L^{-1/2},\ i = 1, \ldots, n. \tag{14}$$

The above formulas (12), (13), and (14) give the idea of the existence of a theory of harmonic analysis parallel to the classical Euclidean case where $L = \Delta = \sum \dfrac{\partial^2}{\partial x_i^2}$.

In the 1980s and with a probabilistic motivation (observe that the Gaussian measure $d\gamma$ is a probability measure), several investigations were carried out on the above concepts. In the 1990s professor E. Fabes got himself engaged on these topics with an harmonic analysis point of view, see [14] and the references therein. Carlos Segovia became interested in this new harmonic analysis, and it would be one of his main fields of work for the rest of his life.

One important issue in the theory of stochastic differential equations is Malliavin's calculus. In particular, a crucial step is the use of Wick polynomials. For short, one could say that Wick polynomials are the natural extension to infinite dimensions of n-dimensional Hermite polynomials. For that reason, results about dimension-free boundedness of operators, like the Riesz transforms defined in (14), became important. In [22] (1996), by using an old friend of Carlos Segovia, an appropriate square function associated to the operators L and ∂_i, it is proved that the Riesz transforms of higher order k, that is,

$$R^\alpha = \partial_i^\alpha \circ L^{-k/2}, \quad \alpha = (\alpha_1, \ldots \alpha_n), \quad \alpha_1 + \cdots + \alpha_n = k,$$

are bounded in $L^p(\mathcal{R}^n, e^{-|x|^2}\, dx)$, $1 < p < \infty$, independently of the dimension n. Previous results about dimension-free boundedness were obtained in [21]; in [20] and [40], by probabilistic methods, and in [45], by using some Calderón-type transference results from the Euclidean case.

Segovia was extremely ill during the years 1999 and 2000. After that period, his health was very weak and he was physically deteriorating; however, his mathematical activity was even stronger than ever. As we said before, he continued contributing on one-sided operators, but his main endeavors were dedicated to this new harmonic analysis. In 2002, in collaboration with L. de Rosa and J.L. Torrea [28], he published a work on the independence of the dimension of the boundedness in $L^p(\mathcal{R}^n, dx)$, $1 < p < \infty$, of the Riesz transforms associated to the Hermite operator $H = -\Delta + |x|^2$, $x \in \mathcal{R}^n$. As in the case of the Ornstein-Uhlenbeck semigroup his old friend the "square function" was the main tool used to prove the result.

In 2004 and in collaboration with E. Harboure, R. Macías, J.L. Torrea, and B. Viviani, Carlos Segovia began a new program related to the Laguerre differential operator. Given a real number $\alpha > -1$, we consider L_α the Laguerre second order differential operator defined by

$$L_\alpha = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y > 0.$$

It is well known that L_α is non-negative and self-adjoint with respect to the Lebesgue measure on $(0, \infty)$. Furthermore, its eigenfunctions are the Laguerre functions $\mathcal{L}_k^\alpha$ defined by

$$\mathcal{L}_k^\alpha(y) = (\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)})^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y),$$

where L_k^α are the Laguerre polynomials of type α, see [81] (p. 100) and [82](p. 7). The orthogonality of Laguerre polynomials with respect to the measure $e^{-y}y^\alpha$ leads to the orthonormality of the family $\{\mathcal{L}_k^\alpha\}_k$ in $L^2((0,\infty), dy)$.

We define the heat-diffusion kernel $W^\alpha(t, y, z)$ for $\alpha > -1$, $t > 0$, $y > 0$, and $z > 0$, as

$$W^\alpha(t, y, z) = \sum_{n=0}^{\infty} e^{-t(n+(\alpha+1)/2)} \mathcal{L}_n^\alpha(y)\, \mathcal{L}_n^\alpha(z),$$

and the heat-diffusion integral $W^\alpha f(t, y)$ as

$$W^\alpha f(t, y) = \int_0^\infty W^\alpha(t, y, z)\, f(z)\, dz.$$

The maximal operator $W^{\alpha,*}$ associated to the heat-diffusion integral $W^\alpha f(t, y)$ is given by

$$W^{\alpha,*} f(y) = \sup_{t>0} |W^\alpha f(t, y)|.$$

Let N_α denote the interval

$$N_\alpha = \begin{cases} \left(\dfrac{2(1+\delta)}{2+\alpha}, \dfrac{2(1+\delta)}{-\alpha}\right) \cap (1,\infty), & \text{if } -1<\alpha<0, \text{ and} \\ \left(\dfrac{2(1+\delta)}{2+\alpha}, \infty\right] \cap (1,\infty] & \text{, if } \alpha \geq 0. \end{cases}$$

We will assume that N_α is not empty. This implies that $1+\delta+\alpha/2>0$ and, if not otherwise stated, we assume this throughout. With this notation, we have the following theorem.

Theorem 6.1 *Let $-1<\alpha<\infty$ and $-1<\delta<\infty$. If $p \in N_\alpha$, then the maximal operator $W^{\alpha,*}$ is of strong type (p,p) with respect to the measure $y^\delta\, dy$, that is,*

$$\int_0^\infty W^{\alpha,*} f(y)^p\, y^\delta\, dy \leq C_{\alpha,\delta,p} \int_0^\infty |f(y)|^p\, y^\delta\, dy$$

holds with a constant $C_{\alpha,\delta,p}$ depending on α, p, and δ only.

The following theorem gives the behavior of $W^{\alpha,*}$ at the end points of N_α.

We denote $a_\alpha = \max(1, \dfrac{2(1+\delta)}{2+\alpha})$ and $b_\alpha = \dfrac{2(1+\delta)}{-\alpha}$ if $-1<\alpha<0$, or $b_\alpha=\infty$ if $\alpha \geq 0$.

Theorem 6.2 *Let $-1<\delta$. Then, at the end points of N_α, we have*

(a) if $-1<\alpha<0$, the operator $W^{\alpha,*}$ is of weak type and not of strong type (b_α, b_α) with respect to the measure $y^\delta\, dy$,

(b) if $\alpha \geq 0$, the operator $W^{\alpha,*}$ is of strong type (∞,∞) with respect to the measure $y^\delta\, dy$,

(c) if $-1<\alpha$ and $a_\alpha = \dfrac{2(1+\delta)}{(2+\alpha)}$, then the operator $W^{\alpha,*}$ is of restricted weak type and not of weak type (a_α, a_α) with respect to the measure $y^\delta\, dy$, and

(d) if $-1<\alpha$ and $a_\alpha=1$, the operator $W^{\alpha,*}$ is of weak type and not of strong type $(1,1)$, with respect to the measure $y^\delta\, dy$.

For the case $\alpha=0$ some previous results had appeared in [80]. The novelty of Theorems 6.1 and 6.2 was that, for negative α, the semigroup was not bounded in all the range of exponents p, $1<p<\infty$, and, for positive α, the family of weights y^δ that were accepted were strictly bigger than the corresponding family of power weights contained in the A_p class of Muckenhoupt. In the case $\delta=0$, the relationship between α and p can be described by Fig. 1. As in other cases in harmonic analysis, sometimes it is more convenient to describe the relation with $\frac{1}{p}$. In this case Fig. 2. appears

The last theorems were published in [37] for the case $\delta=0$ and in [38] for the general case. Segovia was very proud of the development of this harmonic

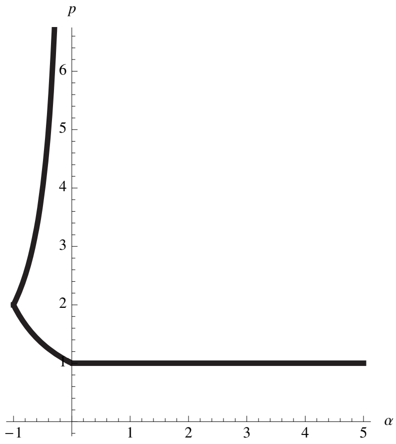

Fig. 1.

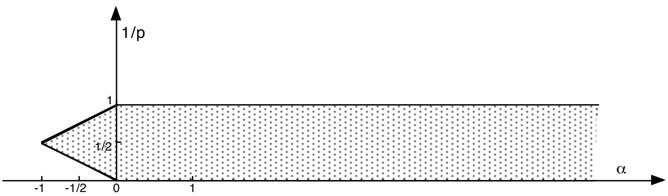

Fig. 2.

analysis in the Laguerre setting. We believe that his ponderings on the subject produced a kind of balsamic effect for him during the last two years of his life. He continued his deep research, and parallel results for the corresponding Riesz transforms were obtained by Professor Carlos Segovia in collaboration with E. Harboure, J.L. Torrea, and B. Viviani; for the interested reader we refer to the paper [30]. Even though his mobility was extremely limited during the final part of his life, Segovia maintained an intense professional traveling activity. His last journey to Spain was in November 2006, where he visited the Universities of Zaragoza and Madrid. The research product of that visit is the article [1], in fact his last paper. In that work it is shown how a result that is proved for a certain family of Laguerre functions can be transferred to another family of Laguerre functions.

Acknowledgement We would like to thank Juan Luis Varona for helping us in the final presentation of this chapter.

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Balls as Subspaces of Homogeneous Type: On a Construction due to R. Macías and C. Segovia

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Summary. In this chapter we give an equivalent point of view for quasi-metric structures on a set X in terms of families of neighborhoods of the diagonal in $X \times X$. We use this approach and the iterative process introduced by R. Macías and C. Segovia in 'A well behaved quasi-distance for spaces of homogeneous type' (Trabajos de Matemática, IAM, 32, 1981, 1–18) in order to show that the balls in the new quasi-distance have a specific regularity at the boundary.

Key words: Quasi-distance, spaces of homogeneous type.

A Carlos Segovia,

...dondequiera que el mar lo haya arrojado,
se aplicará dichoso y desvelado
al otro enigma y a las otras leyes. (JLB)

1 Introduction

Let Q_0 be a cube in $\mathbf{R}^n$. Let Q be another cube in $\mathbf{R}^n$, with sides parallel to those of Q_0, centered at a point of Q_0 and with side length $l(Q)$ less than that of Q_0; then the intersection of both cubes contains another cube with side length at least $\frac{l(Q)}{2^n}$.

This geometric property is actually valid for the family of balls defined by a norm in $\mathbf{R}^n$, not just for $\|x\|_\infty = \sup\{|x_1|, \ldots, |x_n|\}$, for which the balls are the cubes. In fact, take a point x_0 in the Euclidean space $\mathbf{R}^n$ and a positive number R. Consider the Euclidean distance $|x-y|$, or any other norm in $\mathbf{R}^n$. Set $B(x_0, R) = \{x \in \mathbf{R}^n : |x - x_0| < R\}$. Pick a point $x \in B(x_0, R)$ and $0 < r < R$. Then, $B(x + \frac{r}{2}\frac{x_0-x}{|x-x_0|}, r/2) \subset B(x_0, R) \cap B(x, r)$. In fact, for $y \in B(x+\frac{r}{2}\frac{x_0-x}{|x-x_0|}, \frac{r}{2})$ we have that $|y-x| \le \left|y-x-\frac{r}{2}\frac{x_0-x}{|x-x_0|}\right| + \frac{r}{2} < r$. In other words, $y \in B(x, r)$. Also $|y - x_0| \le \left|y - x - \frac{r}{2}\frac{x_0-x}{|x-x_0|}\right| + \left|x - x_0 + \frac{r}{2}\frac{x_0-x}{|x-x_0|}\right| <$

Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_2,

$\frac{r}{2} + \left|\frac{x-x_0}{|x-x_0|}\left(|x-x_0| - \frac{r}{2}\right)\right| = |x-x_0| < R$; then $y \in B(x_0, R)$. Hence, we have that $|B(x_0,R) \cap B(x,r)| \geq a_n r^n$, where a_n depends only on the dimension $n, 0 < r < R$. Here $|E|$ is the Lebesgue measure of the set E. On the other hand, $|B(x_0,R) \cap B(x,r)| \leq |B(x,r)| = w_n r^n$. In other words, there exist constants c_1 and c_2 such that the inequalities

$$c_1 r^n \leq |B(x,r) \cap B(x_0,R)| \leq c_2 r^n$$

hold for every $x \in B(x_0,R)$ and every $0 < r < R$.

This property can be stated by saying that $\{(B(x_0,R), d, \mu) : x_0 \in \mathbf{R}^n,\ R > 0\}$ is a uniform family of n-normal spaces, where d is the restriction of the Euclidean distance to $B(x_0,R)$ and μ is the restriction of Lebesgue measure. Let us point out that s-normality of the balls of a given distance implies that the Hausdorff dimension of open sets is s.

As a consequence of this property of the family of Euclidean balls in $\mathbf{R}^n$, we have that it is a uniform family of spaces of homogeneous type.

It was first observed by Calderón and Torchinsky [1] that this is not the general case, even when the balls are convex sets in $\mathbf{R}^n$.

Let us consider the two-dimensional case $n = 2$. Let $\lambda \geq 1$ be given and set $A_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. The generalized dilations induced on $\mathbf{R}^2$ by A_λ are given by $T_t^\lambda = \begin{pmatrix} t^\lambda & 0 \\ 0 & t \end{pmatrix}$, $t > 0$. If $\|\cdot\|$ is a norm in $\mathbf{R}^2$, then the equation $\|T_t^\lambda x\| = 1$, for $x \in \mathbf{R}^2 - \{0\}$ and λ given has only one solution $t(x) = t_\lambda(x)$. The usual case of parabolic spaces arises when $\lambda = 2$ and the norm is the Euclidean one.

The distance from x to 0 is given by $\frac{1}{t(x)}$. In other words, $\rho(x) = \frac{1}{t(x)}$ if $x \neq 0$ and $\rho(0) = 0$. Notice that ρ depends on λ and on the particular norm $\|\cdot\|$ chosen.

Let us consider the case $\|x\| = \|x\|_1 = |x_1| + |x_2|$. Since $B_\rho(0,r) = T_r(B_\rho(0,1)) = T_r(B_{\|\cdot\|}(0,1))$, we see that $B_\rho(0,r)$ is the rhombus centered at the origin with vertices at the points:

$$T_r((1,0)) = (r^\lambda, 0)\,; T_r((-1,0)) = (-r^\lambda, 0);$$

$$T_r((0,1)) = (0,r)\,\text{and}\,T_r((0,-1)) = (0,-r).$$

For r large consider the ball $B_\rho(v,1) = B_{\|\|_1}(v,1)$ centered at the vertex $v = (r^\lambda, 0)$ of $B_\rho(0,r)$. It is easy to see that the intersection of $B_\rho(0,r)$ and $B_\rho(v,1)$ contains ρ balls of radii at most $r^{1-\lambda}$. Hence, with $\lambda > 1$ there is no chance for the above observed property of the Euclidean balls. Let us denote by $\rho^{1,\lambda}$ this metric. See Fig. 1.

On the other hand, if instead of $\|x\|_1$ we use $\|x\|_\infty = \sup\{|x_1|, |x_2|\}$, we recover for $\rho^{\infty,\lambda}$ that property of Euclidean balls. In fact, the $\rho^{\infty,\lambda}$ balls are now rectangles of the form $[-r^\lambda, r^\lambda] \times [r, r]$, and at no point of the boundary do such small angles occur. Moreover, $\rho^{1,\lambda}$ and $\rho^{\infty,\lambda}$ are equivalent, so the

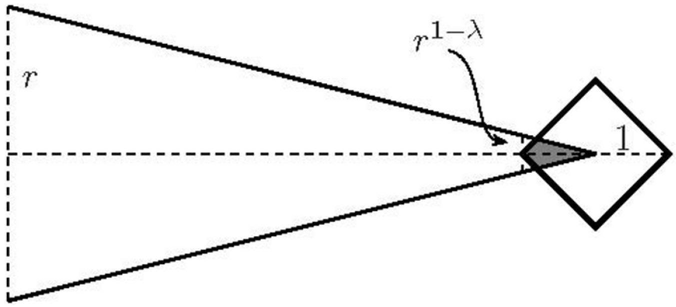

Fig. 1. The situation for $\rho^{1,\lambda}$

property we are looking for is not invariant by equivalence of quasi-distances. See Fig. 2.

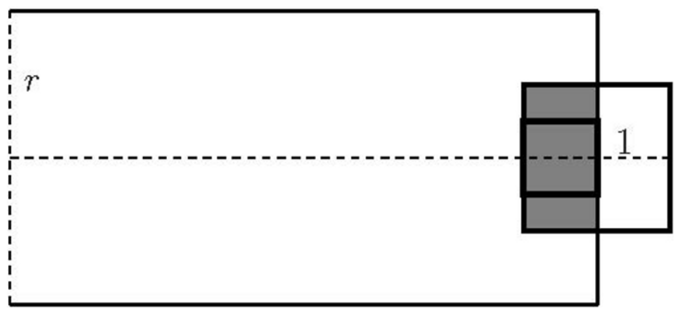

Fig. 2. The situation for $\rho^{\infty,\lambda}$

Issue 32 of the preprint series "Trabajos de Matemática," published in 1981 by the IAM (Buenos Aires), contains the preprint "A well-behaved quasi-distance for spaces of homogeneous type," by Roberto Macías and Carlos Segovia. There the authors show that it is always possible to find an equivalent quasi-distance on a given space of homogeneous type such that balls are spaces of homogeneous type.

The construction of Macías and Segovia is based on an iterative process of composition of the neighborhoods of the diagonal of $X \times X$.

In this chapter we aim to explore the relationship between quasi-distances on X and properties of families of neighborhoods of the diagonal in $X \times X$. We apply the construction of Macías and Segovia to show a somehow stronger version of the uniform regularity of δ-balls for an adequate δ.

2 Quasi-distance on X and diagonal neighborhoods in $X \times X$

Let (X, d) be a quasi-metric space and let K be the triangular constant for d. In other words, d is a nonnegative symmetric function on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$ and $d(x, z) \leq K(d(x, y) + d(y, z))$ for every $x, y, z \in X$. We may assume that the diagonal neighborhoods induced by d are given as a function $V : \mathbf{R}^+ \to \mathcal{P}(X \times X)$, where $\mathbf{R}^+$ is the set of positive real numbers and $\mathcal{P}(X \times X)$ is the set of all subsets of $X \times X$. In fact, define

$$V(r) = \{(x, y) \in X \times X : d(x, y) < r\}.$$

It is easy to prove from the basic properties of d that this family of sets satisfies the following properties:

(a) $V(r_1) \subseteq V(r_2)$ if $r_1 \leq r_2$;
(b) $\cup_{r>0} V(r) = X \times X$;
(c) $\cap_{r>0} V(r) = \Delta$, the diagonal of $X \times X$;
(d) there exists a constant $\mathcal{K}(= 2K)$ such that $V(r) \circ V(r) \subset V(\mathcal{K}r)$ for every $r > 0$. Here $A \circ B$ is the composition $\{(x, z) \in X \times X : \exists y \in X$ such that $(x, y) \in B$ and $(y, z) \in A\}$ of A and B;
(e) each $V(r)$ is symmetric, i.e., $V(r) = V^{-1}(r)$.

The main result of this section is the converse of the preceding remark.

Theorem 1 *Let X be a set and let $V : \mathbf{R}^+ \to \mathcal{P}(X \times X)$ be a given family of subsets of $X \times X$ satisfying properties (a), (b), (c), (d), and (e) above. Then there exists a quasi-distance d on X such that for every $0 < \gamma < 1$ we have that $V(\gamma r) \subseteq V_d(r) \subseteq V(r)$ for every $r > 0$, where $V_d(r) = \{(x, y) : d(x, y) < r\}$.*

Proof Given x and y two points in X, define

$$d(x, y) = \inf\{r : (x, y) \in V(r)\}.$$

From property (b) of the function V we see that d is well defined as a function from $X \times X$ to the set of nonnegative real numbers.

From (c) a couple of the form (x, x) belongs to every $V(r)$, hence $d(x, x) = 0$. On the other hand, (c) also implies that if $x \neq y$, then for some $r > 0$ the pair (x, y) does not belong to $V(r)$. Now, from (a), (x, y) does not belong to any $V(s)$ with $s < r$. Hence, $d(x, y) > 0$. The symmetry of d follows from the symmetry of each $V(r)$. Let us check the triangular inequality. For x, y, z in X and $\varepsilon > 0$ there exist r_1 and r_2 such that

$$r_1 < d(x, y) + \varepsilon \qquad \text{and} \qquad (x, y) \in V(r_1);$$

$$r_2 < d(y, z) + \varepsilon \qquad \text{and} \qquad (y, z) \in V(r_2).$$

Then $(x,z) \in V(r_2) \circ V(r_1)$, which from property (a) of the family V is contained in $V(r_1+r_2) \circ V(r_1+r_2)$. Now, from (d) we have that $V(r_1+r_2) \circ V(r_1+r_2) \subset V(\mathcal{K}(r_1+r_2))$. Hence, (x,z) belongs to $V(\mathcal{K}(r_1+r_2))$, so that, from the very definition of d, we have that

$$d(x,z) \leq \mathcal{K}(r_1+r_2) < \mathcal{K}(d(x,y)+d(y,z)) + 2\mathcal{K}\varepsilon.$$

This proves the triangle inequality for d with $K = \mathcal{K}$ and where d is a quasi-distance on X.

Let us next show that the two families of neighborhoods of Δ, V and V_d, are equivalent. Take any $0 < \gamma < 1$. Assume that $(x,y) \in V(\gamma r)$, then $d(x,y) \leq \gamma r < r$ and $(x,y) \in V_d(r)$. On the other hand, if $(x,y) \in V_d(r)$, then $d(x,y) < r$ so that, for some $s < r$, $(x,y) \in V(s) \subset V(r)$.

We say, as usual, that two quasi-distances d and δ defined on X are equivalent if the function $\frac{d}{\delta}$ is bounded above and below by positive constants on $X \times X - \Delta$. Precisely, $d \sim \delta$ if there exist two constants $0 < c_1 \leq c_2 < \infty$ such that $c_1\delta(x,y) \leq d(x,y) \leq c_2\delta(x,y)$ for every $x, y \in X$. Given V and W two families of neighborhoods of the diagonal, defined as before, we say that V and W are equivalent and we write $V \approx W$ if there exist constants $0 < \gamma_1 \leq \gamma_2 < \infty$ such that

$$V(\gamma_1 r) \subseteq W(r) \subseteq V(\gamma_2 r),$$

for every $r > 0$.

Given a quasi-distance d, we can assign to d a family V_d of neighborhoods of Δ satisfying (a) through (e) in the standard way, $V_d(r) = \{(x,y) : d(x,y) < r\}$.

On the other hand the construction given in Theorem 1 provides a method to assign to every family V of neighborhoods of Δ, satisfying (a) through (e), a quasi-distance d_V such that $V \sim V_{d_V}$.

The next proposition contains basic properties of these equivalences. Let us denote by $\mathcal{V}$ the class of all families satisfying (a) through (e) and by $\mathcal{D}$ the set of all quasi-distances in X.

Proposition 1
(i) For every $V \in \mathcal{V}$ we have that $V \approx V_{d_V}$;
(ii) for every $d \in \mathcal{D}$ we have that $d \sim d_{V_d}$;
(iii) for V and W in $\mathcal{V}$ we have that $V \approx W$ if and only if $d_V \sim d_W$;
(iv) for d and δ in $\mathcal{D}$ we have that $d \sim \delta$ if and only if $V_d \approx V_\delta$.

This proposition shows that we can identify $\mathcal{D}/\sim$ and $\mathcal{V}/\approx$ through

$$\begin{array}{ccc} \mathcal{D} & \overset{V_d}{\underset{d_V}{\rightleftarrows}} & \mathcal{V} \\ \Pi_{\mathcal{D}} \big\downarrow & & \big\downarrow \Pi_{\mathcal{V}} \\ \mathcal{D}/\sim & \overset{H}{\longleftrightarrow} & \mathcal{V}/\approx \end{array}$$

where $H(\overline{d}) = \overline{\overline{V_d}}$ for any $d \in \overline{d}$. Here $\overline{d}$ denotes the $\sim$ class of $d \in \mathcal{D}$ and $\overline{\overline{V_d}}$ denotes the $\approx$ class of $V \in \mathcal{V}$.

Several problems in generalized harmonic analysis are invariant under changes of equivalent quasi-distances. For example, the Hölder–Lipschitz as BMO-type spaces are the same for d and δ if $d \sim \delta$. The Muckenhoupt classes do not change by changing equivalent quasi-metrics, etc. Even some kernels like fractional integrals define equivalent operators. When a particular property of the quasi-distance function becomes relevant for a specific problem in harmonic analysis, the question is whether or not such a property holds for at least one representative of each class in $\mathcal{D}/\sim$.

For example, the well-known result of R. Macías and C. Segovia [2] asserting that each quasi-distance is equivalent to a power of a distance can be written by saying that the mapping

$$D \times \mathbf{R}^+ \overset{J}{\longrightarrow} \mathcal{D}/\sim$$

given by

$$(\rho, \alpha) \longrightarrow J(\rho, \alpha) = \overline{\rho^\alpha}$$

is onto, where D is the family of all distances in X. Notice that J cannot be one to one since $\rho^\alpha = (\rho^{1/2})^{2\alpha}$ and $\rho^{1/2}$ is still a distance if ρ is.

We are interested in finding a family, as large as possible, of measurable subsets Y of X such that (Y, d_Y, μ_Y) is a space of homogeneous type, where d_Y is the restriction of d to Y and μ_Y is the restriction of μ to the measurable subsets of Y. Moreover, from the point of view of its applications in problems of harmonic analysis, we would like to have large families $\mathcal{F}$ of sets Y such that the spaces (Y, d_Y, μ_Y) are uniformly spaces of homogeneous type. If $\mathcal{F}$ is such a family of subsets of X, we briefly say that $\mathcal{F}$ has the u.s.h.t. property. Of course the u.s.h.t. property depends on the distance d and on the measure μ. But it is easy to check that if $\mathcal{F}$ satisfies the u.s.h.t. property with respect to (X, d, μ) and $\delta \sim d$, then $\mathcal{F}$ satisfies the u.s.h.t. property with respect to (X, δ, μ). We shall actually deal with a somehow improved version of u.s.h.t.

3 Regularization of neighborhoods of Δ

The smoothing procedure designed by Macías and Segovia in [2] is based on a self-similarity argument which we proceed to describe. Let d be a quasi-distance on X with constant K. For simplicity of notation, set $U(r) = V_d(r)$. In other words, $U(r) = \{(x, y) : d(x, y) < r\}; r > 0$.

Pick a fixed positive number α less than $\frac{1}{2K}$. We start by the construction of a sequence $U(r, n)$ in the following way:

$$\begin{aligned} U(r, 0) &= U(r) \\ U(r, 1) &= U(ar) \circ U(r) \circ U(ar) \\ U(r, n) &= U(a^n r) \circ U(r, n-1) \circ U(a^n r). \end{aligned}$$

Lemma 1 *For every $r > 0$ and every n we have that*

$$U(r) \subset U(r, n) \subset U(3k^2 r).$$

Proof Since each $U(s)$ contains the diagonal Δ of $X \times X$, then it is clear that $U(r) \subset U(r, n)$ for $n = 0, 1, 2, \ldots$. So we only have to prove that $U(r, n) \subset U(3k^2, r)$. Let $(x, y) \in U(r, n)$. Then, there exists a finite sequence $x = x_0, x_1, x_2, \ldots, x_n, y_n, \ldots, y_2, y_1, y_0 = y$ of points in X such that

$$\begin{aligned} (x_n, y_n) &\in U(r) = U(r, 0), \\ (x_j, x_{j+1}) &\in U(a^{n-j} r), \text{and} \\ (y_j, y_{j+1}) &\in U(a^{n-j} r),\ j = 0, 1, \ldots, n. \end{aligned}$$

Let us now estimate $d(x, y)$ by repeated use of the triangle inequality,

$$\begin{aligned} d(x, y) = d(x_0, y_0) &\leq K^2 [d(x_0, x_n) + d(x_n, y_n) + d(y_n, y_0)] \\ &\leq K^2 \left[\textstyle\sum_{i=0}^{n-1} d(x_i, x_{i+1}) K^{n-i} + d(x_n, y_n) + \sum_{i=0}^{n-1} d(y_i, y_{i+1}) K^{n-i}\right] \\ &< K^2 \left[\textstyle\sum_{i=0}^{n-1} a^{n-i} r K^{n-i} + r + \sum_{i=0}^{n-1} a^{n-i} r K^{n-i}\right] \\ &= r K^2 \left[1 + 2 \textstyle\sum_{i=0}^{n-1} (aK)^{n-i}\right] \\ &< 3K^2 r. \end{aligned}$$

The inequality $d(d, y) < 3K^2 r$ proves that $(x, y) \in U(3K^2 r)$, and the lemma is proved.

Given a positive r, define

$$V(r) = \bigcup_{n=0}^{\infty} U(r, n).$$

Lemma 2 *The family $V(r)$ is equivalent to the family $U(r)$ and satisfies properties (a) through (e) in Section 1.*

Proof Let us start by noticing that from Lemma 1, for every $r > 0$ we have $U(r) \subset V(r) \subset U(3k^2 r)$. If $r_1 \leq r_2$, then $U(a^j r_1) \subseteq U(a^j r_2)$. Hence $U(r_1, n) \subseteq U(r_2, n)$ and $V(r_1) \subseteq V(r_2)$, which proves (a). Properties (b) and (c) for $V(r)$ follow from properties (b) and (c) for $U(r)$ and from Lemma 1. The symmetric of $V(r)$ follows from the symmetric of $U(r)$. Let us finally check that $V(r)$ satisfies property (d). In fact, from Lemma 1 we have that

$$V(r) \circ V(r) \subset U(3K^2 r) \circ U(3K^2 r) \subset U(6K^3 r) \subset V(bK^3 r),$$

which is (d) for $V(r)$ with $\mathcal{K} = 6K^3$.

Let $\delta = d_V$. Then $\delta \sim d$. The basic scheme of the procedure is

$$d \to U = V_d \to V \to d_V = \delta.$$

Here the middle arrow is the regularization procedure of Macías and Segovia and the other two are the canonical mappings.

Since the regularization procedure produces equivalent neighborhood systems, we obtain a new quasi-distance δ which is equivalent to d.

4 The main result

In this section we aim to prove the following result.

Theorem 2 *Let (X, d) be a quasi-metric space. Then there exist a quasi-distance δ on X and a positive constant γ such that*

(i) $\delta \sim d$;
(ii) the δ-balls are open sets;
(iii) for every $x \in X$, every choice of r and R with $0 < r < R$, and every $x \in B_\delta(x_0, R)$, there exists a point $\xi \in X$ such that

$$B(\xi, \gamma_r) \subset B_\delta(x, R) \cap B(y, r).$$

We are using the notation B_δ for the δ-balls and B for the balls defined by the original quasi-distance d on X. We may think of B as the testing balls for B_δ, the tested set. Hence, the family of d-balls can be changed to the family of balls corresponding to any other quasi-distance equivalent to d, in particular by the δ-balls.

Let us first notice that it is enough to prove the theorem for the case of a distance ρ on X instead of a general quasi-distance d. In fact, given d on X there exists ρ a distance on X and $\alpha \geq 1$ such that $d \sim \rho^\alpha$. Let us assume that we know the result for ρ. In other words, there exist a quasi-distance δ_ρ

and $\gamma_\rho > 0$ with $\delta_\rho \sim \rho$, the δ_ρ-balls open sets and, for every $x \in X$, every $0 < s < S$, and every $y \in B_{\delta_\rho}(x, S)$, we have that, for some $\xi \in X$,

$$B_\rho(\xi, \gamma_\rho s) \subset B_{\delta_\rho}(x, S) \cap B_\rho(y, s).$$

Take $\delta = \delta_\rho^\alpha \sim \rho^\alpha \sim d$. Since $B_\delta(z, r) = B_{\delta_\rho}(z, r^{1/\alpha})$ we see that δ-balls are open sets. Notice also that $B_{\rho^\alpha}(z, r) = B_\rho(Z, r^{1/\alpha})$. Take now $x \in X, 0 < r < R$, and $y \in B_\delta(x, R)$. Then, with $s = r^{1/\alpha}$, $S = R^{1/\alpha}$ we have that $y \in B_{\delta_\rho}(x, S)$. Hence, from our assumption, we get

$$\begin{aligned} B(\xi, c\gamma_\rho^\alpha r) &\subset B_{\rho^\alpha}(\xi, \gamma_\rho^\alpha r) = B_\rho(\xi, \gamma_\rho s) \\ &\subseteq B_{\delta_\rho}(x, S) \bigcap B_\rho(y, s) \subseteq B_\delta(x, R) \bigcap B(y, \bar{c}r). \end{aligned}$$

From now on we shall assume that d is a distance on X, i.e., $K = 1$. One more reduction of Theorem 2 is in order. Assume that we can prove that there exists a positive λ such that for every distance d on X there exists a quasi-distance δ on X satisfying (i), (ii), and (iii) with $R = 1$. Then the theorem follows. In fact, if d is a distance, so is μd for $\mu > 0$. Since λ does not depend on the given distance, the scaling argument is clear. To obtain our main result, we only have to prove the following statement.

Proposition 2 *There exists a positive constant λ such that, for every metric space (X, d), there exists a quasi-distance δ on X, equivalent to d, such that the δ-balls are open sets and for every $x \in X, 0 < r < 1$ and $y \in B_\delta(x, 1)$ there exists $\xi \in X$ such that*

$$B(\xi, \lambda r) \subset B_\delta(x, 1) \cap B(y, r).$$

For a metric space the result of Lemma 1 reads

$$U(r) \subset U(r, n) \subset U(3r),$$

for every $r > 0$ and every $n \in \mathbf{N}$ if $U(r) = \{(x, y) : d(x, y) < r\}$. For fixed $n \in \mathbf{N}$ the family $\mathcal{U}_n = \{U(r, n) : r > 0\}$ satisfies properties (a) through (e). Then

$$\delta_n(x, y) = \inf\{r > 0 : (x, y) \in U(r, n)\}$$

is a quasi-distance on X, equivalent to d. Moreover, since $U(r) \subset U(r, n) \subset U(3r)$ for every n, the equivalence constants are independent of n. Also, the triangular constants are uniformly bounded above. In fact, if $x, y, z \in X$ and $\varepsilon > 0$, pick r_1 and r_2 such that $r_1 < \delta_n(x, y) + \varepsilon$ and $r_2 < \delta_n(y, z) + \varepsilon$ with $(x, y) \in U(r_1, n)$ and $(y, z) \in U(r_2, n)$. Thus,

$$
\begin{aligned}
(x,y) \in U(r_1,n) \circ U(r_2,n) &\subseteq U(r_1+r_2,n) \circ U(r_1+r_2,n) \\
&\subseteq U(3(r_1+r_2)) \circ U(3(r_1+r_2)) \\
&\subseteq U(6(r_1+r_2)),
\end{aligned}
$$

which proves that $\delta_n(x,z) \leq 6(\delta_n(x,y) + \delta_n(y,z))$. Since $U(r,n) \subset V(r)$ for every $r > 0$ and every $n \in \mathbf{N}$, we have that $\delta_n(x,y) \geq \delta(x,y)$. On the other hand, since $V(r) = \bigcup_n U(r,n)$, we have the pointwise convergence of $\delta_n(x,y)$ to $\delta(x,y)$.

Lemma 3 *For every $x \in X$ and every $r > 0$ we have that*

$$B_{\delta_n}(x,r) = \{y : (x,y) \in U(r,n)\}$$

and that

$$B_\delta(x,r) = \{y : (x,y) \in V(r)\} = \bigcup_n B_{\delta_n}(x,r).$$

Proof Applying the argument used in Section 3 for the case of a distance d, we can take the number a to be $\frac{1}{4}$. Hence, $U(r,0) = U(r) = \{(x,y) : d(x,y) < r\}$ and $U(r,n) = U(\frac{r}{4^n}) \circ U(r,n-1) \circ U(\frac{r}{4^n})$ for $n \in \mathbf{N}$.

If $y \in B_{\delta_n}(x,r)$, then $\delta_n(x,y) < r$, so that $(x,y) \in U(s,n)$ for some $s < r$ and $(x,y) \in U(r,n)$. Take now $(x,y) \in U(r,n)$, then there exists a chain of points in X

$$x_0 = x, x_1, x_2, \ldots, x_{n-1}, x_n; y_n, y_{n-1}, \ldots, y_2, y_1, y_0 = y$$

such that $d(x_n,y_n) < r$; $d(x_{n-j+1}, x_{n-j}) < r4^{-j}$; and $d(y_{n-j+1}, y_{n-j}) < r4^{-j}$ for $j = 0,1,\ldots,n$.

Since we have a finite number of strict inequalities, for some $s < r$ we must certainly also have that $d(x_n,y_n) < s$; $d(x_{n-j+1},x_{n-j}) < s4^{-j}$; and $d(y_{n-j+1},y_{n-j}) < s4^{-j}$; for $j = 0,1,2,\ldots,n$. This fact proves that $(x,y) \in U(s,n)$ for some $s < r$, then $\delta_n(x,y) \leq s < r$. Hence, $y \in B_{\delta_n}(x,r)$.

Notice also that since d is a distance, the sets $U(r)$ and $U(r,n)$ are open sets in $X \times X$. The same is true for each $V(r)$. Hence, their sections B_{δ_n} and B_δ are open sets in X.

The proof of Proposition 2 is then reduced to the proof of the next result.

Proposition 3 *Let (X,d) be a metric space. Then, for every $n \in \mathbf{N}$, for every $i \in \{1,\ldots,n\}$, for every $x \in X$ and for every $y \in B_{\delta_n}(x,1)$, there exists a point $\xi \in X$ such that*

$$B(\xi, 4^{-i-1}) \subseteq B_{\delta_n}(x,1) \cap B(y,4^{-i}).$$

Also, $B(y,4^{-i}) \subseteq B(\xi, 4^{-i+1})$.

Proof Since $y \in B_{\delta_n}(x,1)$, then $\delta_n(x,y) < 1$. From the definition of δ_n we know that there exists $s \in (0,1)$ such that $(x,y) \in U(s,n)$. In other words, there exists $s \in (0,1)$ and a finite sequence in X

$$C : x = x_0, x_1, x_2, \ldots, x_{n-1}, x_n; y_n, y_{n-1}, \ldots, y_2, y_1, y_0 = y$$

such that $d(x_n, y_n) < s$, $d(x_{n-j+1}, x_{n-j}) < s4^{-j}$ and $d(y_{n-j+1}, y_{n-j}) < s4^{-j}$ for $j = 0, 1, 2, \ldots, n$.

Fix now i in $\{1, \ldots, n\}$. The proposition is valid with $\xi = y_{n-i}$ if we prove

(A) $B(y_{n-i}, 4^{-i-1}) \subseteq B_{\delta_n}(x,1)$;

(B) $B(y, 4^{-i}) \subseteq B(y_{n-i}, 4^{-i+1})$;

(C) $B(y_{n-i}, 4^{-i-1}) \subseteq B(y, 4^{-i})$.

Proof of (A): Take $z \in B(y_{n-i}, 4^{-i-1})$. From Lemma 3, in order to show that $z \in B_{\delta_n}(x,1)$, we only need to check that (z,x), or (x,z), is an element of $U(1,n)$. Let us use some points in the chain C to produce another chain C' joining x to z,

$$C' : x = x_0, x_1, x_2, \ldots, x_{n-1}, x_n; y_n, y_{n-1}, \ldots, y_{n-i}, \underbrace{z, z, \ldots, z}_{n-i}.$$

Let us see that C' is an admissible chain to show that $(x,z) \in U(1,n)$,

$$\begin{array}{l} d(x_n, y_n) < s, \\ d(x_{n-j+1}, x_{n-j}) < s4^{-j},\ j = 0, \ldots, n \\ d(y_{n-j+1}, y_{n-j}) < s4^{-j},\ j = 0, \ldots, i \\ d(y_{n-i}, z) < 4^{-i-1},\ \text{since}\, z \in B(y_{n-i}, 4^{-i-1}) \\ d(z,z) = 0. \end{array}$$

Since $s < 1$ we have that C' satisfies the required condition for $(x,z) \in U(1,n)$.

Proof of (B): Take $z \in B(y, 4^{-i})$. Then

$$\begin{aligned} d(z, y_{n-i}) &\le d(z, 0) + d(y, y_{n-i}) \\ &< 4^{-i} + d(y_0, y_{n-i}) \\ &\le 4^{-i} + \textstyle\sum_{j=i+1}^{n} d(y_{n-j+1}, y_{n-j}) \\ &< 4^{-i} + s \textstyle\sum_{j=i+1}^{n} 4^{-j} < 4^{-i+1}. \end{aligned}$$

Proof of (C): Take $z \in B(y_{n-i}, 4^{-i-1})$. Then

$$d(z,y) \le d(z,y_{n-i}) + d(y_{n-i},y)$$
$$< 4^{-i-1} + \tfrac{s}{3}4^{-i} < 4^{-i}.$$

Let us finally point out that, even when the preceding proof is essentially that given in [2], Theorem 2 is giving more information than doubling. If, for example (X,d,μ) is an s-normal space, then the family of δ-balls is a uniform family of s-normal subspaces.

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Some Aspects of Vector-Valued Singular Integrals

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Summary. Let A, B be Banach spaces and $1 < p < \infty$. T is said to be a (p, A, B)-Calderón–Zygmund type operator if it is of weak type (p, p), and there exist a Banach space E, a bounded bilinear map $u : E \times A \to B$, and a locally integrable function k from $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ into E such that

$$Tf(x) = \int u(k(x, y), f(y))dy$$

for every A-valued simple function f and $x \notin supp\ f$.

The study of the boundedness of such operators between spaces of vector-valued functions under the classical assumption of the kernel is established.

Given a bounded operator T from $L^p_A(\mathbb{R}^n)$ to $L^p_B(\mathbb{R}^n)$, and $\mathcal{L}(A)$ and $\mathcal{L}(B)$-valued functions b_1 and b_2, we define the commutator

$$T_{b_1,b_2}(f) = b_2T(f) - T(b_1f)$$

for any A-valued simple function f. The boundedness of commutators of (p, A, B)-Calderón–Zygmund type operators and operator-valued functions in a space of bounded mean oscillation (BMO), under certain commuting properties on the couple (b_1, b_2), are also analyzed.

Key words: Vector-valued singular integrals, commutators, operator-valued functions, Hardy spaces, bounded mean oscillation.

To Carlos, a good mathematician and friend, always willing to discuss life and mathematics.

1 Introduction and notation

Two of the basic operators in harmonic analysis are the well-known "Hilbert transform" and the "Hardy–Littlewood maximal operator." The Hardy–Littlewood maximal function is defined in $\mathbb{R}^n$ by

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_3,

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int |f(y)|dy,$$

where Q is a cube in $\mathbb{R}^n$ and $|Q|$ stands for the Lebesgue measure of the cube.

The Hilbert transform is defined in $\mathbb{R}$ in several equivalent ways:

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}dy,$$

$$Hf(x) = \lim_{t\to 0} Q_t * f(x),$$

where Q_t is the conjugate Poisson kernel, $Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+t^2}$, or

$$(Hf\hat{)}(\xi) = -isgn(\xi)\hat{f}(\xi).$$

Their boundedness properties in the Lebesgue spaces are rather well known, and their possible generalizations to higher dimensions, other spaces or their vector-valued analogues have been a source of creativity for mathematicians for decades.

These operators are known to be of weak type $(1,1)$,

$$|\{x : |Hf(x)| > \lambda\}| \le C\frac{\|f\|_1}{\lambda} \tag{1}$$

$$|\{x : |Mf(x)| > \lambda\}| \le C\frac{\|f\|_1}{\lambda} \tag{2}$$

and of strong type (p,p)

$$\|Hf\|_p \le C\|f\|_p, \quad 1 < p < \infty, \tag{3}$$

$$\|Mf\|_p \le C\|f\|_p, \quad 1 < p \le \infty. \tag{4}$$

The endpoint results for $p = 1$ and $p = \infty$ for the Hilbert transform lead to two very important spaces in harmonic analysis, namely $BMO(\mathbb{R})$ and $H^1(\mathbb{R})$.

$BMO(\mathbb{R})$ consists of locally integrable functions such that $\sup_Q osc_p(f,Q) < \infty$ for some (or equivalently for all) $0 < p < \infty$, where

$$osc_p(f,Q) = (\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx)^{1/p}$$

and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$ for a cube Q in $\mathbb{R}$.

The space can also be described by means of the sharp maximal function of f ([FS]),

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx.$$

Of course, $f \in BMO(\mathbb{R})$ if and only if $f^{\#} \in L^\infty(\mathbb{R})$ and the "norm" in BMO is then given by $\|f^{\#}\|_\infty$.

Let us also recall that $BMO(\mathbb{R})$ is the dual space of $H^1(\mathbb{R})$ ([FS, C]), where H^1 is the Hardy space defined in terms of atoms, that is, the space of integrable functions $f = \sum_k \lambda_k a_k$, $\lambda_k \in \mathbb{R}$, $\sum_k |\lambda_k| < \infty$ and where a_k belong to $L^\infty(\mathbb{R})$, $supp(a_k) \subset Q_k$ for some cube Q_k, $\int_{Q_k} a(x)dx = 0$, and $|a(x)| \leq \frac{1}{|Q_k|}$. The norm is now given by the infimum of $\sum_k |\lambda_k|$ over all possible decompositions of f.

With these definitions out of the way, we can now mention that the Hilbert transform maps $H^1(\mathbb{R})$ into $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$, i.e.,

$$\|Hf\|_{BMO} \leq C\|f\|_\infty, \tag{5}$$

$$\|Hf\|_1 \leq C\|f\|_{H^1}. \tag{6}$$

There is another rather relevant connection between BMO and the commutator between multiplication operators and singular integrals. It was first shown by Coifman, Rochberg, and Weiss ([CRW]) that, denoting $H_b(f) = bH(f) - H(bf)$, one has

$$b \in BMO(\mathbb{R}) \text{ if and only if } H_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \text{ is bounded.} \tag{7}$$

There have been several directions in which results from (1) through (7) have been extended and developed. One approach, which goes back to the early 1950s, was the consideration of convolution operators with kernels similar in higher dimensions. The big contribution to the general theory of operators that share similar properties with the Hilbert transform is due to the work of A. Calderón and A. Zygmund (see [CZ]).

It took some time for people to isolate the properties that were needed in the kernel in order to apply similar techniques to other classes of operators.

After some preliminary stages (see [S], [H]) in the case of convolution kernels, the theory was then extended to non-convolution operators (see [CM] or [J]).

Following Coifman and Meyer we shall say that T is a generalized Calderón–Zygmund operator if it has the following properties:

(i) T is bounded on $L^2(\mathbb{R}^n)$,

(ii) there exists a locally integrable function k from $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ into $\mathbb{C}$ such that

$$Tf(x) = \int k(x,y)f(y)dy \tag{8}$$

for every bounded and compactly supported function f and $x \notin supp\ f$,

(iii) there exists $\varepsilon > 0$ such that the kernel satisfies

$$|k(x,y)| \leq \frac{C}{|x-y|}, \tag{9}$$

$$|k(x,y) - k(x',y)| \leq C\frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}}, \quad |x-y| \geq 2|x-x'|, \tag{10}$$

$$|k(x,y)-k(x,y')| \le C\frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}}, \quad |x-y| \ge 2|y-y'|. \tag{11}$$

In general, for generalized Calderón–Zygmund operators one does not have the identity

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} k(x,y)f(y)dy, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{12}$$

The operators for which (12) happens to hold true were called Calderón–Zygmund type singular integrals.

For such operators one defines the corresponding maximal operator

$$T^*(f)(x) = \sup_{\varepsilon>0} |\int_{|x-y|>\varepsilon} K(x,y)f(y)dy|.$$

Let us collect in the following theorem the boundedness properties of such operators.

Theorem 1.1 *Let T be a Calderón–Zygmund type singular operator defined on $\mathbb{R}^n$. Then*

$$|\{x : |Tf(x)| > \lambda\}| \le C\frac{\|f\|_1}{\lambda},$$

$$\|Tf\|_1 \le C\|f\|_{H^1},$$

$$\|Tf\|_p \le C\|f\|_p, \quad 1<p<\infty,$$

$$\|Tf\|_{BMO} \le C\|f\|_\infty,$$

$$|\{x : |T^*f(x)| > \lambda\}| \le C\frac{\|f\|_1}{\lambda},$$

$$\|T^*f\|_p \le C\|f\|_p \quad 1<p<\infty.$$

It is known (see [D] Theorem 5.12) that to get the boundedness and the weak type $(1,1)$ for such operator conditions, (9), (10), and (11) can be replaced by the Hörmander type ones

$$\int_{|x-y|\ge 2|x-x'|} |k(x,y)-k(x',y)| \le C, \tag{13}$$

$$\int_{|x-y|\ge 2|y-y'|} |k(x,y)-k(x,y')| \le C. \tag{14}$$

Another direction, which we shall consider throughout this paper, is the vector-valued consideration of the problem.

There are two different points of view. One is to consider sequences of functions (f_j), rather than a single one, and ask ourselves whether inequalities like

$$\|(\sum_j |Hf_j|^r)^{1/r}\|_p \le C\|(\sum_j |f_j|^r)^{1/r}\|_p \tag{15}$$

hold true for some values $0 < p, r < \infty$.

Another one is to analyze whether inequalities like

$$\|(\sum_j |T_j f|^r)^{1/r}\|_p \leq C\|f\|_p \tag{16}$$

hold true for some values $0 < p, r < \infty$, where T_j are Calderón–Zygmund operators with some assumptions.

These results, nowadays, fit in a theory of general vector-valued functions, when allowing either the function or the kernel to take values in Banach spaces.

A systematic study and its applications to Littlewood–Paley theory or maximal functions were first started in the work of Benedek, Calderón, and Panzone ([BCP]) and then continued by Rubio de Francia, Ruiz, and Torrea ([RRT]).

The vector-valued theory also has some applications in a more abstract setting and interplays with the geometry of Banach spaces when analyzing the properties on the Banach space for several classical results to hold true in the Banach-valued setting.

One of the most important vector-valued extensions is the following: Let X be a Banach space and consider $f = \sum_{k=1}^m \phi_k x_k$, where $\phi_k \in L^p(\mathbb{R})$ and $x_k \in X$. We can now define the vector-valued Hilbert transform

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy = \sum_{k=1}^m H(\phi_k) x_k \in X.$$

The main question is to know whether the operator, defined on $L^p(\mathbb{R}) \otimes X$, extends continuously to $L^p_X(\mathbb{R})$. A positive answer for $X = \ell^r$ answers the question in (15). In this direction it is not difficult to see that for $X = \ell^1(\mathbb{N})$ the operator $H \otimes Id$ is not bounded on $L^2_{\ell^1}(\mathbb{R})$. However, it follows from a general ℓ^2-valued extension result due to Marcinkiewicz and Zygmund (see [GR], Theorem 2.7) that (15) holds for the case $X = \ell^2(\mathbb{N})$. The description of the property on the space X for the boundedness of the vector-valued Hilbert transform goes back to the work of Burkholder and Bourgain (see [Bu1, Bo1]) who proved to be equivalent to the boundedness of the unconditional martingale differences transform and, from then on called the Unconditional Martingale Differences (UMD) property.

From this point of view, and realizing that the boundedness on L^2 is equivalent to the one in any L^p for $1 < p < \infty$, one can say that (15) holds for $1 < p, r < \infty$ because ℓ_r has the UMD property in the case $1 < r < \infty$.

The UMD property became rather relevant, due to the interplay of the Hilbert transform with other areas whenever functions were allowed to take values in Banach spaces.

Another example of the use of vector-valued analysis in connection with classical operators comes from the Banach-lattice-valued extension of the

Hardy–Littlewood maximal function ([GMT]). A Banach function space L is said to have the Hardy–Littlewood property if

$$M_L(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(x)|dx,$$

where the supremum is taken in the lattice structure. Several characterizations and results concerning this property were achieved in [GMT, GMT2].

I would like to point out here that Carlos Segovia became interested in the interplay between the geometry of Banach spaces and vector-valued analysis, and we refer the reader to his work in [HMST] for a combination of techniques from A_p theory, harmonic analysis, and the use of the Hardy–Littlewood property in Banach lattices.

As for the Hilbert transform, given a Banach space X and a Calderón–Zygmund operator T with scalar-valued kernel k, one defines the vector-valued extension

$$T_X f(x) = \sum_{k=1}^{m} T(\phi_k)x_k \in X$$

for $f = \sum_{k=1}^m \phi_k x_k$, where $\phi_k \in L^p(\mathbb{R})$ and $x_k \in X$.

The next generalization in the theory comes from the fact that the expression $\int k(x,y)f(y)dy$ for vector-valued functions f also makes sense for operator-valued kernels, with the obvious interpretation.

Let now A and B be Banach spaces and denote by $\mathcal{L}(A,B)$ the space of bounded linear operators from A to B. Following Segovia and Torrea (see [ST1]), we shall say that T is an $\mathcal{L}(A,B)$-Calderón–Zygmund type operator if it has the following properties:

(i) $T : L^p(\mathbb{R}^n, A) \to L^p(\mathbb{R}^n, B)$ is bounded for some $1 < p < \infty$ and

(ii) there exists a locally integrable function k from $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ into $\mathcal{L}(A,B)$ such that

$$Tf(x) = \int k(x,y)f(y)dy$$

for every A-valued bounded and compactly supported function f and $x \notin$ *supp* f.

Several properties can be imposed on these kernels in order to obtain the corresponding boundedness properties in different spaces:

$$\int_{|x-y|\geq 2|x-x'|} \|k(x,y) - k(x',y)\|dy \leq C, \qquad (H_y)$$

$$\int_{|x-y|\geq 2|y-y'|} \|k(x,y) - k(x,y')\|dx \leq C, \qquad (H_x)$$

$$\|k(x,y) - k(x',y)\| \leq C\frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}}, \quad |x-y| \geq 2|x-x'|, \qquad (CZ_y)$$

$$\|k(x,y) - k(x,y')\| \leq C\frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}}, \quad |x-y| \geq 2|y-y'|. \qquad (CZ_x)$$

Note that (CZ_y) (resp. (CZ_x)) implies (H_y) (resp. (H_x)).

Throughout the chapter we shall work on $\mathbb{R}^n$ endowed with the Lebesgue measure dx and use the notation $|A| = \int_A dx$. Given a Banach space $(X, \|\cdot\|)$ and $1 \leq p < \infty$, we shall denote by $L^p_X(\mathbb{R}^n)$ the space of Bochner p-integrable functions endowed with the norm $\|f\|_{L^p_X} = (\int_{\mathbb{R}^n} \|f(x)\|^p dx)^{1/p}$ and by $L^\infty_c(\mathbb{R}^n, X)$ the closure of the compactly supported functions in $L^\infty_X(\mathbb{R}^n)$.

We use the notation Mf and $f^\#$ for the Hardy–Littlewood maximal function and the sharp maximal function of f, i.e.,

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q \|f(x)\|dx,$$

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q \|f(x) - \frac{1}{|Q|}\int_Q f(y)dy\|dx).$$

We shall denote $M_q(f) = M(\|f\|^q)^{1/q}$ for $1 < q < \infty$. We shall use the following results shown by Fefferman and Stein (see [FS, GR]):

$$f^\#(x) \approx \sup_{x\in Q} \inf_{c_Q \in X} \frac{1}{|Q|}\int_Q \|f(x) - c_Q\|dx. \qquad (17)$$

If $M(f) \in L^{p_0}$ for some $0 < p_0 < \infty$, then for $1 < p < \infty$,

$$\|f\|_{L^p_X(\mathbb{R}^n)} \leq C\|f^\#\|_{L^p(\mathbb{R}^n)}. \qquad (18)$$

We write $H^1_X(\mathbb{R}^n)$ for the Hardy space defined by X-valued atoms, consisting of integrable functions $f = \sum_k \lambda_k a_k$, where $\lambda_k \in \mathbb{R}$, $\sum_k |\lambda_k| < \infty$, and a_k belong to $L^\infty_c(\mathbb{R}^n, X)$, $supp(a_k) \subset Q_k$ for some cube Q_k, $\int_{Q_k} a(x)dx = 0$, and $\|a(x)\| \leq \frac{1}{|Q_k|}$.

Now $BMO_X(\mathbb{R}^n)$ stands for the space of locally integrable functions such that $\sup_Q osc_p(f,Q) < \infty$ for some (or equivalently for all) $1 \leq p < \infty$, where

$$osc_p(f,Q) = (\frac{1}{|Q|}\int_Q \|f(x) - f_Q\|^p dx)^{1/p}$$

and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$ for a cube Q in $\mathbb{R}^n$.

The interested reader should be aware that the duality H^1_X and BMO_{X^*} and the formulations of the spaces of H^1_X or BMO_X by means of the Hilbert transform are no longer true for infinite-dimensional Banach spaces X. One needs either the RNP on X^* or the UMD property in X (see [RRT, Bl1, Bl2]) for the corresponding results to hold true.

The following theorem exhibits the final achievements in the case $\mathcal{L}(A,B)$-valued kernels and goes back to the work of Benedek, Calderón, and Panzone for convolution kernels and to the work of Rubio, Ruiz, and Torrea for non-convolution ones.

Theorem 1.2 *([BCP], [RRT]) Let A, B be Banach spaces and let T be an $\mathcal{L}(A,B)$-Calderón–Zygmund type operator satisfying (H_x) and (H_y). Then*

(i) T is weak type (1,1), i.e.,

$$|\{x : \|Tf(x)\|_B > \lambda\}| \leq C\frac{\|f\|_{L^1_A}}{\lambda};$$

(ii) T is bounded from $L^q_A(\mathbb{R}^n)$ to $L^q_B(\mathbb{R}^n)$.

Throughout the literature many results appeared in connection with the boundedness of commutators of Calderón–Zygmund type operators and multiplication by a function b given by $T_b(f) = bT(f) - T(bf)$ on many different spaces, in the weighted and vector-valued settings. Let me now mention some result on operator-valued singular integrals, which is one of the important contributions of Carlos Segovia to this theory.

Theorem 1.3 *([ST1, Theorem 1]) Let A, B be Banach spaces and let T be an $\mathcal{L}(A,B)$-valued Calderón–Zygmund type operator such that the kernel satisfies (CZ_y). Let $\ell \to \bar{\ell}$ be a correspondence from $\mathcal{L}(A)$ to $\mathcal{L}(B)$ such that*

$$\tilde{\ell}T(f)(x) = T(\ell f)(x) \tag{19}$$

and

$$k(x,y)\ell = \tilde{\ell}k(x,y). \tag{20}$$

If b is $\mathcal{L}(A)$-valued, $b \in BMO_{\mathcal{L}(A)}(\mathbb{R}^n)$, and $\tilde{b} \in BMO_{\mathcal{L}(B)}(\mathbb{R}^n)$, then

$$T_b(f) = bT(f) - T(bf)$$

is bounded from $L^p_A(\mathbb{R}^n) \to L^p_B(\mathbb{R}^n)$ for all $1 < p < \infty$.

Endpoint estimates for the commutator was also a topic that attracted several people in different directions (see [CP, HST, PP, P1, PT2]). The endpoint estimates of the previous operator-valued result were later studied by E. Harboure, C. Segovia, and J.L. Torrea (see Theorem A and Theorem 3.1 in [HST]) when b was assumed to be scalar valued. We will not go in this direction, but from their results one concludes that non-constant scalar-valued BMO functions do not define bounded commutators from $L^\infty_c(\mathbb{R}^n, A)$ to $BMO_B(\mathbb{R}^n)$ when kernels of the Calderón–Zygmund type operators are $\mathcal{L}(A,B)$-valued and also that, in general, T_b does not map $H^1_A(\mathbb{R}^n)$ into $L^1_B(\mathbb{R}^n)$.

We shall present a proof of these last theorems in a slightly more general situation and we recommend that the interested reader look for their applications to Maximal functions, Littlewood Paley theory, and other topics in [RRT] and the work by Carlos in [ST1] (see also Chapter V in [GR] and [ST2, ST3, ST4, ST5] for similar results in related operators and applications).

As usual, we denote λQ for a cube centered at x_Q (center of Q) and with side length $\lambda\ell(Q)$, and C will denote a constant that may vary from line to line.

2 Theorems and proofs

Although the vector-valued theory has been developed for operator-valued kernels, we shall see in what follows that most of the techniques used there can also be applied in a slightly more general situation.

Definition 2.1 *Let A, B be Banach spaces and $1 < p < \infty$. We say that T is a (p, A, B)-Calderón–Zygmund type operator if it is of weak type (p, p),*

$$|\{x : \|Tf(x)\|_B > \lambda\}| \leq C\frac{\|f\|^p_{L^p_A}}{\lambda^p}, \tag{21}$$

and there exist a Banach space E, a bounded bilinear map $u : E \times A \to B$, and a locally integrable function k from $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ into E such that

$$Tf(x) = \int u(k(x, y), f(y))dy \tag{22}$$

for every A-valued simple function f and $x \notin supp\ f$.

We then say that T has kernel k with respect to u.

We shall also use the notation from the previous section (H_x), (H_y), (CZ_x) and (CZ_y) also when replacing the norm in $\mathcal{L}(A, B)$ by the norm in E.

As in the scalar case, the basic ingredients in our proofs will be the Calderón–Zygmund decomposition and the Kolmogorov inequality.

Lemma 2.2 *([GR], Theorem 1.2) Let ϕ be a non-negative integrable function and let $\lambda > 0$. There exists a sequence of disjoint dyadic cubes $\{Q_j\}$ such that*

$$\phi(x) \leq \lambda, a.a. \ \ x \notin \cup_j Q_j,$$

$$| \cup_j Q_j| \leq \frac{\|\phi\|_1}{\lambda},$$

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} \phi \leq 2^n\lambda.$$

We shall need the following lemma due to Kolmogorov.

Lemma 2.3 *Let $1 \leq q < p$ and let Q be a cube. If T is any operator of weak type (p, p), then*

$$(\frac{1}{|Q|}\int_Q \|Tf(x)\|^q_B dx)^{1/q} \leq C(\frac{1}{|Q|}\int_Q \|f(x)\|^p_A dx)^{1/p}.$$

Proof Use the weak type (p, p) condition to estimate

$$\begin{aligned}\int_Q \|Tf(x)\|^q dx &= q\int_0^\infty t^{q-1}|\{x\in Q : \|Tf(x)\| > t\}|dt \\ &\le q\int_0^\infty t^{q-1}\min\{|Q|, C^p\frac{\|f\|_p^p}{t^p}\}dt \\ &= q\int_0^{C\frac{\|f\|_p}{|Q|^{1/p}}} t^{q-1}|Q|dt + q\int_{C\frac{\|f\|_p}{|Q|^{1/p}}}^\infty C^p\frac{\|f\|_p^p}{t^{p-q+1}}dt \\ &\le C|Q|^{1-q/p}\|f\|^q_{L^p_A}. \square\end{aligned}$$

Proposition 2.1 *Let A, B be Banach spaces. Let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u which satisfies (H_x). Then*

(i) T maps $H^1_A(\mathbb{R}^n)$ to $L^1_B(\mathbb{R}^n)$.

(ii) T is weak type (1,1).

Proof (i) It suffices to show that $\|T(a)\|_{L^1_B} \le C$ for any A-valued atom.

Let a be an integrable function supported on a cube Q and with mean value zero with $\|a(x)\| \le \frac{1}{|Q|}$. Using Lemma 2.3 for $q = 1$,

$$\int_{2Q} \|Ta(x)\|dx \le C|Q|^{1-\frac{1}{p}}\|a\|_{L^p_A} \le C|Q|(\frac{1}{|Q|}\int_Q \|a(x)\|^p dx)^{1/p} \le C.$$

To analyze the other part, we shall use (H_x). For each $x \notin 2Q$, then

$$Ta(x) = \int_Q u(k(x,y), a(y))dy = \int_Q u(k(x,y) - k(x,x_Q), a(y))dy.$$

$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2Q} \|Ta(x)\|_B dx &\le \int_{\mathbb{R}^n\setminus 2Q}\int_Q \|u(k(x,y) - k(x,x_Q), a(y))\|dydx \\ &\le \int_Q (\int_{|x-y|>2|y-x_Q|} \|u\|\|k(x,y) - k(x,x_Q)\|dx)\|a(y)\|dy \\ &\le C\|a\|_{L^1_A} \le C.\end{aligned}$$

(ii) Let $f \in L^1_A(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n, A)$ and let $\lambda > 0$. Now apply Lemma 2.2 to $\|f\|$ and denote $\Omega = \cup_j Q_j$. This allows us to decompose $f = g + b$, where

$$g = \sum_j (\frac{1}{|Q_j|}\int_{Q_j} f)\chi_{Q_j} + f\chi_{\mathbb{R}^n\setminus\Omega} \tag{23}$$

and

$$b = \sum_j (f - \frac{1}{|Q_j|}\int_{Q_j} f)\chi_{Q_j}. \tag{24}$$

Observe that $\|g(x)\| \le C\lambda$ a.e. and $\|g\|_{L^1_A} \le \|f\|_{L^1_A}$. Therefore,

$$
\begin{aligned}
|\{x : \|Tf(x)\|_B > \lambda\}| &\leq |\{x : \|Tg(x)\|_B > \lambda/2\}| + |\{x : \|Tb(x)\|_B > \lambda/2\}| \\
&\leq \frac{C}{\lambda^p} \|Tg\|^p_{L^p_B} + |\{x : \|Tb(x)\|_B > \lambda/2\}| \\
&\leq \frac{C}{\lambda^p} \|g\|^p_{L^p_A} + |\{x : \|Tb(x)\|_B > \lambda/2\}| \\
&\leq \frac{C}{\lambda} \int \|g(x)\|_A dx + |\{x : \|Tb(x)\|_B > \lambda/2\}| \\
&\leq \frac{C}{\lambda} \int \|f(x)\|_A dx + |\{x : \|Tb(x)\|_B > \lambda/2\}|.
\end{aligned}
$$

As above, if a is an integrable function supported on a cube Q and with mean value zero, then

$$\int_{\mathbb{R}^n \setminus 2Q} \|Ta(x)\|_B dx \leq C \|a\|_{L^1_A}.$$

In particular, denoting $b_j = (f - \frac{1}{|Q_j|} \int_{Q_j} f)\chi_{Q_j}$, one has

$$\int_{\mathbb{R}^n \setminus 2Q_j} \|Tb_j(x)\| dx \leq C \int_{Q_j} \|b_j(y)\| dy \leq 2C \int_{Q_j} \|f(y)\| dy.$$

Using $f\chi_\Omega \in L^p_A(\mathbb{R}^n)$ one easily gets that $\sum_j b_j$ converges in $L^p_A(\mathbb{R}^n)$. This implies that

$$|\{x : \|Tb(x) - \sum_{j=1}^N Tb_j(x)\| > \varepsilon\}| \leq C \frac{\|b - \sum_{j=1}^N b_j\|^p_{L^p_A}}{\varepsilon^p}$$

for any $\varepsilon > 0$, which shows that $\|Tb(x)\|_B \leq \sum_j \|Tb_j(x)\|_B$ a.e.

This allows us to conclude that, for $\tilde{\Omega} = \cup_j 2Q_j$,

$$
\begin{aligned}
|\{x : \|Tb(x)\|_B > \lambda/2\}| &\leq |\tilde{\Omega}| + \frac{2}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \|Tb(x)\| dx \\
&\leq C|\Omega| + \frac{2}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus 2Q_j} \|Tb_j(x)\| dx \\
&\leq C|\Omega| + \frac{2}{\lambda} \sum_j \int_{Q_j} \|f(x)\|_A dx \\
&\leq C \frac{\|f\|_{L^1_A}}{\lambda}. \square
\end{aligned}
$$

Proposition 2.2 *Let A, B be Banach spaces. Let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u which satisfies (H_y). Then T maps $L^\infty_c(\mathbb{R}^n, A)$ to $BMO_B(\mathbb{R}^n)$.*

Proof Let f be an A-valued simple function and let Q be a cube in $\mathbb{R}^n$. Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$. Now write $c_Q = (T(f_1))_Q - T(f_2)(x_Q)$, where x_Q is the center of Q,

$$Tf(x) - c_Q = T(f_1)(x) - (T(f_1))_Q + T(f_2)(x) - T(f_2)(x_Q).$$

Clearly, from Lemma 2.3,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \|T(f_1)(x) - (T(f_1))_Q\|dx &\leq \frac{2}{|Q|}\int_Q \|T(f_1)(x)\|dx \\ &\leq C(\frac{1}{|Q|}\int_Q \|f(x)\|^p dx)^{1/p} \leq C\|f\|_\infty.\end{aligned}$$

On the other hand, for $x \in Q$ and $y \notin 2Q$ one has $|x-y| \geq 2|x - x_Q|$, which implies

$$T(f_2)(x) - T(f_2)(x_Q) = \int_{(2Q)^c} u(k(x,y) - k(x_Q,y), f_2(y))dy.$$

Now the assumption (H_y) gives

$$\begin{aligned}\|T(f_2)(x) - T(f_2)(x_Q)\| &\leq \|u\| \int_{(2Q)^c} \|k(x,y) - k(x_Q,y)\|\|f(y)\|dy \\ &\leq C\|f\|_\infty \int_{|x-y|\geq 2|x-x_Q|} \|k(x,y) - k(x_Q,y)\|dy \\ &\leq C\|f\|_\infty.\end{aligned}$$

Therefore, $\frac{1}{|Q|}\int_Q \|T(f)(x) - c_Q\|dx \leq C\|f\|_\infty$, which implies that $\|Tf\|_{BMO} \leq C\|f\|_\infty$ for any $f \in L^\infty_c(\mathbb{R}^n, A)$. $\square$

Using Propositions 2.1 and 2.2 together with interpolation, one obtains the following result.

Theorem 2.4 *Let A, B be Banach spaces and let T be a (p, A, B)-Calderón–Zygmund type operator with kernel satisfying (H_x) and (H_y).*

Then T extends to a bounded operator from $L^q_A(\mathbb{R}^n)$ to $L^q_B(\mathbb{R}^n)$ for all $1 < q < \infty$.

Proof We can use the interpolation results (see [Bl3, BX])

$$[H^1_A(\mathbb{R}^n), L^\infty_c(\mathbb{R}^n, A)]_\theta = L^p_A(\mathbb{R}^n), \frac{1}{p} = 1 - \theta,$$

and

$$[L^1_B(\mathbb{R}^n), BMO_B(\mathbb{R}^n)]_\theta = L^p_B(\mathbb{R}^n), \frac{1}{p} = 1 - \theta.\square$$

We shall now see that a condition slightly weaker than (CZ_y) allows us to get a certain pointwise majoration which also gives the strong type in all $1 < q < \infty$ for (p, A, B)-Calderón–Zygmund type operators.

Proposition 2.3 *Let A, B be Banach spaces and let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u. Assume the kernel satisfies the existence of $q > 1$ and a sequence (A_j) such that $\sum_{j=1}^{\infty} A_j < \infty$, where*

$$\sup_{|x-x'|\leq R} (2^j R)^{n/q} \Big(\int_{2^j R \leq |x-y| < 2^{j+1}R} \|k(x,y) - k(x',y)\|^{q'} dy\Big)^{1/q'} \leq A_j. \quad (25)$$

Then, for $s = \max\{p, q\}$,

$$T^{\#} f(x) \leq C M_s(f)(x)) \quad (26)$$

for any A-valued bounded and compactly supported function f.

Proof Let $x \in \mathbb{R}^n$ and let Q be a cube centered at x and radius R. Given a compactly supported and bounded function f, we write $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$.

Note that (22) gives for any $z \in Q$

$$Tf_2(z) = \int_{|y-x|>2R} u(k(z,y), f_2(y))dy.$$

Denote $c_Q = T(f_2)(x)$. Therefore, for $x' \in Q$,

$$\begin{aligned}
\|Tf_2(x') - c_Q\| &= \Big\| \int_{|y-x|>2R} u(k(x',y) - k(x,y), f_2(y))dy \Big\| \\
&= \Big\| \int_{|y-x|>2R} u(k(x',y) - k(x,y), f_2(y))dy \Big\| \\
&\leq \|u\| \int_{|y-x|>2R} \|k(x',y) - k(x,y)\| \|f_2(y)\| dy \\
&\leq \|u\| \sum_{j=1}^{\infty} \int_{2^j R \leq |y-x| < 2^{j+1}R} \|k(x',y) - k(x,y)\| \|f_2(y)\| dy \\
&\leq \|u\| \sum_{j=1}^{\infty} (2^j R)^{-n/q} A_j \Big(\int_{2^j R \leq |y-x| < 2^{j+1}R} \|f_2(y)\|^q dy\Big)^{1/q} \\
&\leq C\|u\| \sum_{j=1}^{\infty} A_j \Big(\frac{1}{2^{(j+1)n} R^n} \int_{|y-x| < 2^{j+1}R} \|f(y)\|^q dy\Big)^{1/q} \\
&\leq C\|u\| \Big(\sum_j A_j\Big) M_q f(x).
\end{aligned}$$

Hence,

$$\frac{1}{|Q|}\int_Q \|Tf(x') - c_Q\|dx' \le \frac{1}{|Q|}\int_Q \|Tf_1(x')\|dx' + C\|u\|M_q f(x).$$

Using Lemma 2.3 one gets

$$\frac{1}{|Q|}\int_Q \|Tf_1(x')\|dx' \le C(\frac{1}{|Q|}\int_{2Q} \|f(y)\|^p dy)^{1/p} \le CM_p(f)(x).$$

Therefore, $(Tf)^{\#}(x) \le C\Big(M_p(f)(x) + M_q(f)(x)\Big) \le CM_s(f)(x)$. □

3 Commutators

Let us define a notion which is needed for our purposes.

Definition 3.1 *Let T be a bounded operator from $L^p_A(\mathbb{R}^n)$ to $L^p_B(\mathbb{R}^n)$, and let b_1 and b_2 be $\mathcal{L}(A)$- and $\mathcal{L}(B)$-valued functions, respectively. Define*

$$T_{b_1,b_2}(f) = b_2T(f) - T(b_1 f)$$

for any A-valued simple function f.

We shall be using the following basic assumptions for our general version of the commutator theorem.

Definition 3.2 *Let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u and let b_1 and b_2 be $\mathcal{L}(A)$- and $\mathcal{L}(B)$-valued functions respectively. We shall say that (b_1, b_2) has the commuting properties $(CP)_1$ and $(CP)_2$ if*

$(CP)_1$ $\quad b_2(z)u(k(x, y), a) = u(k(x, y), b_1(z)a), \quad x, y, z \in \mathbb{R}^n, x \ne y.$

$(CP)_2$ $\quad (b_2)_Q T(a\chi_A)(x') = T((b_1)_Q a\chi_A)(x')$ *for any Q cube, $A \subset Q$, $a \in A$, and $x' \in Q$.*

We would like to point out that $(CP)_1$ produces the following cancellation property.

Lemma 3.3 *Let (b_1, b_2) satisfy $(CP)_1$, let Q, Q' be cubes in $\mathbb{R}^n$, and let f_1 and f_2 be simple A-valued with supp$f_1 \subset Q'$ and supp$f_2 \subset (Q')^c$. Then*

$$(b_2)_Q T(f_1)(x) = T((b_1)_Q f_1)(x), \quad x \in (Q')^c \tag{27}$$

$$(b_2)_Q T(f_2)(x) = T((b_1)_Q f_2)(x), \quad x \in Q'. \tag{28}$$

Proof We prove only (28); the other case follows in a similar way.

Recall that if $F \in L^1(\mathbb{R}^n, X)$ and $\Phi \in \mathcal{L}(X)$ for a given Banach space, then $\Phi(\int F(x)dx) = \int \Phi F(x)dx$. Hence,

$$\begin{aligned}(b_2)_Q T(f_2)(x) &= (b_2)_Q(\int_{(Q')^c} u(k(x,y), f_2(y))dy \\ &= \int_{(Q')^c} (b_2)_Q u(k(x,y), f_2(y))dy \\ &= \int_{(Q')^c} (\frac{1}{|Q|}\int_Q b_2(z)dz)u(k(x,y), f_2(y))dy \\ &= \int_{(Q')^c} (\frac{1}{|Q|}\int_Q b_2(z)u(k(x,y), f_2(y)dz)dy \\ &= \int_{(Q')^c} (\frac{1}{|Q|}\int_Q u(k(x,y), b_1(z)f_2(y))dz)dy \\ &= \int_{(Q')^c} u(k(x,y), (\frac{1}{|Q|}\int_Q b_1(z)dz)f_2(y))dy \\ &= T((b_1)_Q f_2)(x).\Box\end{aligned}$$

Example 3.4 *Let $S_1 : A \to B$, $S_2 : B \to B$, and $S : A \to B$ be bounded operators such that $S_2 S = S S_1$. Let $u : \mathbb{C} \times A \to B$ be given by $(\lambda, a) \to \lambda S a$ and k be a scalar-valued kernel. Define $T_S(\sum_{j=1}^N \phi_k a_k) = \sum_{j=1}^N T(\phi_k)S(a_k)$, where T is the scalar-valued Calderón–Zygmund operator with kernel k.*

If $b_1(x) = b(x)S_1$ and $b_2(x) = b(x)S_2$ for some scalar-valued functions $b(x)$, then (b_1, b_2) has the commuting properties $(CP)_1$ and $(CP)_2$. In this case one has

$$T_{b_1,b_2}(f)(x) = b(x)S_2(T_S f(x)) - T_S(bS_1 f)(x).$$

Example 3.5 *Let A be a Banach space and let $u : A^* \times A \to \mathbb{C}$ be given by $u(a^*, a) = \langle a^*, a\rangle$. Let k be an A^*-valued function and let T be a Calderón–Zygmund operator from $L^p_A(\mathbb{R})$ into $L^p(\mathbb{R})$ with kernel k with respect to u.*

If $b_1(x) = b(x)Id_A$ and $b_2(x) = b(x)$ for a scalar-valued b, then (b_1, b_2) has the commuting properties $(CP)_1$ and $(CP)_2$. In this case one has

$$T_{b_1,b_2}(f)(x) = b(x)Tf(x) - T(bf)(x).$$

Example 3.6 *Let $b(x) \in A^*$ and let $u : \mathbb{C} \times A \to A$ given by $(\lambda, a) \to \lambda a$. Let $k(x,y)$ be a scalar-valued function and T a Calderón–Zygmund operator with kernel k and denote $T_A = T \otimes Id_A$. If $b_1(x)(a) = b_2(x)(a) = \langle b(x), a\rangle a_0$ for a fixed $a_0 \in A$, then (b_1, b_2) has the commuting properties $(CP)_1$ and $(CP)_2$. In this case one has*

$$T_{b_1,b_2}(f) = (\langle b, T_A(f)\rangle - T(\langle b, f\rangle)a_0.$$

Let us start with the basic facts which follow from $(CP)_1$ and $(CP)_2$.

Lemma 3.7 *Let (b_1, b_2) satisfy $(CP)_1$ and $(CP)_2$, let Q be a cube in $\mathbb{R}^n$, and let f be simple A-valued. Then*

$$(b_2)_Q T(f)(x) = T((b_1)_Q f)(x)\, x \in Q. \tag{29}$$

Proof Take $f_1 = f\chi_Q$ and $f_2 = f - f_1$ and invoke Lemma 3.3 to obtain $(b_2)_Q T(f_2)\chi_Q = T((b_1)_Q f_2)\chi_Q$. Now $(CP)_2$ gives $(b_2)_Q T(f_1)\chi_Q = T((b_1)_Q f_1)\chi_Q$. □

Another useful lemma which is essentially included in [HST] is the following.

Lemma 3.8 *Let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u and which satisfies the assumption (CZ_y). If f is compactly supported A-valued with supp$f \subset (2Q)^c$, then*

$$\|T(f)(x) - T(f)(x')\| \leq C\|u\| \frac{|x-x'|^\varepsilon}{\ell(Q)^\varepsilon} \sum_{j=2}^{\infty} \frac{2^{-j\varepsilon}}{|Q_j|} \int_{Q_j} \|f(y)\| dy, \qquad x, x' \in Q, \tag{30}$$

where we are denoting $Q_j = 2^j Q$.

Proof Note that if $x, x' \in Q$, then

$$T(f)(x) - T(f)(x') = \int_{(2Q)^c} u(k(x,y) - k(x',y), f(y))dy.$$

$$\begin{aligned}
\|T(f)(x) - T(f)(x')\| &\leq \|u\| \int_{(2Q)^c} \|k(x,y) - k(x',y)\| \|f(y)\| dy \\
&\leq C\|u\| |x-x'|^\varepsilon \int_{(2Q)^c} \frac{\|f(y)\|}{|x-y|^{n+\varepsilon}} dy \\
&\leq C\|u\| |x-x'|^\varepsilon \sum_{j=1}^{\infty} \int_{Q_{j+1}-Q_j} \frac{\|f(y)\|}{|x-y|^{n+\varepsilon}} dy \\
&\leq C\|u\| |x-x'|^\varepsilon \sum_{j=2}^{\infty} \frac{1}{\ell(Q_j)^{n+\varepsilon}} \int_{Q_j} \|f(y)\| dy \\
&\leq C\|u\| \frac{|x-x'|^\varepsilon}{\ell(Q)^\varepsilon} \sum_{j=2}^{\infty} 2^{-j\varepsilon} \frac{1}{|Q_j|} \int_{Q_j} \|f(y)\| dy. \square
\end{aligned}$$

We shall now present our result on commutators. It is just an adaptation of the proof in [ST1] under slightly weaker assumptions.

Proposition 3.1 *Let $b_1 \in BMO(\mathbb{R}^n, \mathcal{L}(A))$, $b_2 \in BMO(\mathbb{R}^n, \mathcal{L}(B))$, and $1 < p < \infty$. Let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u which satisfies (CZ_y). Let (b_1, b_2) satisfy $(CP)_1$ and $(CP)_2$. Then, for any $1 < q < p < s < \infty$, there exists $C_{q,s} > 0$ such that*

$$T_{b_1,b_2}(f)^{\#}(x) \leq C_{q,s}(\|b_2\|_{BMO} M_q(Tf)(x) + \|b_1\|_{BMO} M_s(f)(x)).$$

Proof Let f be a simple E-valued function. Let Q be a cube and denote $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. Put $c_Q = T([(b_1)_Q - b_1]f_2)(x_Q)$.

For each $x \in Q$ one has, applying Lemma 3.7,

$$T_{b_1,b_2}f(x) = b_2Tf(x) - T(b_1f)(x)$$

$$= [b_2 - (b_2)_Q]Tf(x) + T([(b_1)_Q - b_1]f)(x)$$

$$= [b_2 - (b_2)_Q]Tf(x) + T([(b_1)_Q - b_1]f_1)(x) + T([(b_1)_Q - b_1]f_2)(x).$$

Hence,

$$T_{b_1,b_2}f(x) - c_Q = \sum_{i=1}^{3} \sigma_i(x),$$

where

$$\sigma_1(x) = [b_2 - (b_2)_Q]Tf(x),$$

$$\sigma_2(x) = T([(b_1)_Q - b_1]f_1)(x)$$

and

$$\sigma_3(x) = T([(b_1)_Q - b_1]f_2)(x) - T([(b_1)_Q - b_1]f_2)(x_Q).$$

Observe that for any $q > 1$ and $1/q + 1/q' = 1$ we can write

$$\frac{1}{|Q|}\int_Q \|\sigma_1(x)\|dx \leq osc_{q'}(b_2, Q)(\frac{1}{|Q|}\int_Q \|Tf(x)\|^q dx)^{1/q}.$$

For any $s > p > q_1 > 1$ one can write, for $1/s + 1/r = 1/p$, from Lemma 2.3,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \|\sigma_2(x)\|dx &\leq (\frac{1}{|Q|}\int_Q \|T([(b_1)_Q - b_1]f_1)(x)\|^{q_1} dx)^{1/q_1} \\ &\leq C(\frac{1}{|Q|}\int_Q \|(b_1 - (b_1)_Q)f_1(x)\|^p dx)^{1/p} \\ &\leq Cosc_r(b_1, Q)(\frac{1}{|Q|}\int_Q \|f(x)\|^s dx)^{1/s}.\end{aligned}$$

Using Lemma 3.8, and taking into account that $\|(b_1)_Q - (b_1)_{2Q}\| \leq Cosc_{q_1}(b_1, 2Q)$, we also can estimate, for a given $s > 1$,

$$
\begin{aligned}
\|\sigma_3(x)\| &\leq C\sum_{j=2}^{\infty} 2^{-j\varepsilon}\frac{1}{|Q_j|}\int_{Q_j}\|(b_1(y)-(b_1)_Q)f(y)\|dy \\
&\leq C\sum_{j=2}^{\infty} 2^{-j\varepsilon}(\frac{1}{|Q_j|}\int_{Q_j}\|b_1(y)-(b_1)_Q\|^{s'}dy)^{1/s'}(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^{s}dy)^{1/s} \\
&\leq C\sum_{j=2}^{\infty} 2^{-j\varepsilon}(\sum_{k=2}^{j} osc_{s'}(b_1,Q_k))(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^{s}dy)^{1/s} \\
&\leq C\sup_{j\geq 2}(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^{s}dy)^{1/s}\Big(\sum_{j=2}^{\infty} 2^{-j\varepsilon}(\sum_{k=2}^{j} osc_{q'}(b_1,Q_k))\Big) \\
&\leq C\|b_1\|_{BMO}\sup_{j\geq 2}(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^{s}dy)^{1/s}\sum_{j} j2^{-j\varepsilon} \\
&\leq C\|b_1\|_{BMO}M_s(f)(x).
\end{aligned}
$$

Hence, combining the previous estimates, one obtains

$$
T_{b_1,b_2}(f)^{\#}(x) \leq C\|b_2\|_{BMO}M_q(Tf)(x) + C\|b_1\|_{BMO}M_s(f)(x)).\square
$$

Theorem 3.9 *Let $b_1 \in BMO(\mathbb{R}^n, \mathcal{L}(A))$ and $b_2 \in BMO(\mathbb{R}^n, \mathcal{L}(B))$. Let T be a (p, A, B)-Calderón–Zygmund type operator with kernel k with respect to u which satisfies (H_x) and (CZ_y) and let (b_1, b_2) satisfy $(CP)_1$ and $(CP)_2$. Then T_{b_1,b_2} is bounded from $L^q_A(\mathbb{R}^n)$ to $L^q_B(\mathbb{R}^n)$ for any $1 < q < \infty$.*

Proof First use Theorem 2.4 to obtain that T is an (s, A, B)-Calderón–Zygmund type operator for all $1 < s < p$. In fact, T is bounded from $L^r_A(\mathbb{R}^n)$ to $L^r_A(\mathbb{R}^n)$ for all $1 < r < \infty$. Use, for a given $1 < q < \infty$, Proposition 3.1 for $1 < q_1, s_1 < q$, which, by combining the boundedness of M_{q_1} and M_{s_1} in $L^q_B(\mathbb{R}^n)$ and the boundedness of T from $L^q_A(\mathbb{R}^n)$ to $L^q_B(\mathbb{R}^n)$, imply that

$$
\begin{aligned}
\|T_{b_1,b_2}f\|_{L^q_B(\mathbb{R}^n)} &\leq C\|T_{b_1,b_2}(f)^{\#}\|_{L^q_B(\mathbb{R}^n)} \\
&\leq C(\|b_2\|_{BMO}\|M_{q_1}(Tf)\|_{L^q} + C\|b_1\|_{BMO}\|M_{s_1}(f)\|_{L^q}) \\
&\leq C(\|b_2\|_{BMO}\|T\|\|f\|_{L^q} + C\|b_1\|_{BMO}\|f\|_{L^q}).\square
\end{aligned}
$$

Acknowledgement

Partially supported by Proyecto MTM 2005-08350 and MTM2008-04594.

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Products of Functions in Hardy and Lipschitz or BMO Spaces

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Summary. We define as a distribution the product of a function (or distribution) h in some Hardy space $\mathcal{H}^p$ with a function b in the dual space of $\mathcal{H}^p$. Moreover, we prove that the product $b \times h$ may be written as the sum of an integrable function with a distribution that belongs to some Hardy–Orlicz space, or to the same Hardy space $\mathcal{H}^p$, depending on the values of p.

Key words: Orlicz spaces, atomic decomposition, Hardy spaces, local Hardy spaces, bounded mean oscillation (BMO).

This paper is dedicated to the memory of Carlos Segovia.

1 Introduction

For p and p' two conjugate exponents, with $1 < p < \infty$, when we consider two functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n) = (L^p(\mathbb{R}^n))^*$, their product fg is integrable, which means in particular that their pointwise product gives rise to a distribution. When $p = 1$, the right substitute to Lebesgue spaces is, for many problems, the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, whose dual is the space $\mathrm{BMO}(\mathbb{R}^n)$. So one may ask what is the right definition of the product of $h \in \mathcal{H}^1(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. In this context, the pointwise product is not integrable in general. In order to get a distribution, one has to define the product in a different way. This question has been considered by the first author in a joint work with T. Iwaniec, P. Jones, and M. Zinsmeister in [BIJZ]. This chapter explores the same problem in different spaces.

The duality bracket $\langle b, h\rangle$ may be written through the almost everywhere approximation of the factor $b \in \mathrm{BMO}(\mathbb{R}^n)$,

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_4,

$$\langle b, h\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^n} b_k(x)h(x)\,dx\ , \tag{1}$$

where b_k is a sequence of bounded functions, which is bounded in the space $\mathrm{BMO}(\mathbb{R}^n)$ and converges to b almost everywhere. For example, we can choose

$$b_k(x) = \begin{cases} k & \text{if} \quad k \le b(x) \\ b(x) & \text{if } -k \le b(x) \le \quad k \\ -k & \text{if} \qquad\quad b(x) \le -k \end{cases}. \tag{2}$$

We then define the product $b \times h$ as the distribution whose action on the test function φ in the Schwartz class, that is, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, is given by

$$\langle b \times h, \varphi\rangle := \langle b\varphi, h\rangle. \tag{3}$$

We use the fact that the multiplication by φ is a bounded operator on $\mathrm{BMO}(\mathbb{R}^n)$. So the right-hand side makes sense in view of the duality $\mathcal{H}^1$-BMO. Alternatively, the Schwartz class is contained in the space of multipliers of $\mathrm{BMO}(\mathbb{R}^n)$, which have been studied and characterized, see [S] and the discussion below. It follows from (1), used with the sequence b_k given in (2), that the distribution $b \times h$ is given by the function bh whenever this last one is integrable.

A more precise description of products $b \times h$ has been given in [BIJZ]. Namely, all such distributions are sums of a function in $L^1(\mathbb{R}^n)$ and a distribution in a Hardy–Orlicz space $\mathcal{H}^{\Phi}_w$, where w is a weight which allows a smaller decay at infinity and Φ is given below. We will consider a slightly different situation by replacing the space $\mathrm{BMO}(\mathbb{R}^n)$ by the smaller space $\mathfrak{bmo}(\mathbb{R}^n)$, defined as the space of locally integrable functions b such that

$$\sup_{|B|\le 1} \left(\frac{1}{|B|}\int_B |b - b_B|dx\right) < \infty \quad \text{and} \quad \sup_{|B|\ge 1} \left(\frac{1}{|B|}\int_B |b|dx\right) < \infty. \tag{4}$$

Here B varies among all balls of $\mathbb{R}^n$ and $|B|$ denotes the measure of the ball B. Also, b_B is the mean of b on the ball B. Recall that the BMO condition reduces to the first one, but for all balls and not only for balls B such that $|B| < 1$. We clearly have $\mathfrak{bmo} \subset \mathrm{BMO}$.

We have the following, which is new compared to [BIJZ].

Theorem 1.1 *For h a function in $\mathcal{H}^1(\mathbb{R}^n)$ and b a function in $\mathfrak{bmo}(\mathbb{R}^n)$, the product $b \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$b \times h \in L^1(\mathbb{R}^n) + \mathcal{H}^{\Phi}_*(\mathbb{R}^n). \tag{5}$$

$\mathcal{H}^{\Phi}_*(\mathbb{R}^n)$ is a variant of the Hardy–Orlicz space related to the Orlicz function

$$\Phi(t) := \frac{t}{\log(e+t)}, \tag{6}$$

which is defined in Section 3. It contains $\mathcal{H}^{\Phi}(\mathbb{R}^n)$ and is contained in the weighted Hardy–Orlicz space that has been considered in [BIJZ] for the general case of $f \in \mathrm{BMO}$.

The aim of this chapter is to give some extensions of the previous situation. Indeed, (3) makes sense in other cases. First, the space $\mathfrak{bmo}$ is the dual of the local Hardy space, as proved by Goldberg [G], who introduced it. So it is natural to extend the previous theorem to functions h in this space, which we do. Next, we can consider the Hardy space $\mathcal{H}^p(\mathbb{R}^n)$, for $p < 1$, and its dual the homogeneous Lipschitz space $\dot{\Lambda}_\gamma(\mathbb{R}^n)$, with $\gamma := n(\frac{1}{p} - 1)$. Indeed, a function in the Schwartz class is also a multiplier of the Lipschitz spaces. Our statement is particularly simple when b belongs to the inhomogeneous Lipschitz space $\Lambda_\gamma(\mathbb{R}^n)$.

Theorem 1.2 *Let $p < 1$ and $\gamma := n(\frac{1}{p}-1)$. Then, for h a function in $\mathcal{H}^p(\mathbb{R}^n)$ and b a function in $\Lambda_\gamma(\mathbb{R}^n)$, the product $b \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$b \times h \in L^1(\mathbb{R}^n) + \mathcal{H}^p(\mathbb{R}^n). \tag{7}$$

Again, the space $\Lambda_\gamma(\mathbb{R}^n)$ is the dual of the local version of the Hardy space $\mathcal{H}^p(\mathbb{R}^n)$. We will adapt the theorem to h in this space.

Let us explain the presence of two terms in the two previous theorems. The product looses the cancellation properties of the Hardy space, which explains the term in L^1. Once we have subtracted some function in L^1, we recover a distribution of a Hardy space. For $p = 1$, there is a loss, due to the fact that a function in $\mathfrak{bmo}$ is not bounded, but uniformly in the exponential class on each ball of measure 1. This explains why we do not find a function in $\mathcal{H}^1$, but in the Hardy–Orlicz space.

As we will see, the proof uses a method that is linear in b, not in h. As in the case $\mathcal{H}^1$-BMO (see [BIJZ]), one would like to know whether the decomposition of $b \times h$ as a sum of two terms can be obtained through linear operators, but we are very far from being able to answer this question.

All this study is reminiscent of problems related to commutators with singular integrals, or Hankel operators. In particular, such products arise when developing commutators between the multiplication by b and the Hilbert transform and looking separately at each term. It is well known that the commutator $[b, H]$ maps $\mathcal{H}^p(\mathbb{R})$ into $\mathcal{H}^1_{\mathrm{weak}}(\mathbb{R})$ for b in the Lipschitz space $\Lambda_\gamma(\mathbb{R})$ (see [J]), which means that there are some cancellations between terms, compared to our statement, which is the best possible for each term separately. One can also consider products of holomorphic functions in the corresponding spaces when $\mathbb{R}^n$ is replaced by the torus, considered as the boundary of the unit disc. Statements and proofs are much simpler, and there are converse statements, see [BIJZ] for the case $p = 1$, and also [BG], where the problem is treated in general for holomorphic functions in Hardy–Orlicz spaces in convex domains of finite type in $\mathbb{C}^n$. These results allow us to characterize the classes of symbols for which Hankel operators are bounded from some Hardy–Orlicz space larger than $\mathcal{H}^1$ into $\mathcal{H}^1$.

Another possible generalization deals with spaces of homogeneous type instead of $\mathbb{R}^n$. Since the seminal work of Coifman and Weiss [CW1, CW2], it has been a paradigm in harmonic analysis that this is the right setting for developing the Calderón–Zygmund theory. The contribution of Carlos Segovia, mainly in collaboration with Roberto Macías, has been fundamental in developing a general theory of Hardy and Lipschitz spaces. We will rely on their work in the last section, when explaining how properties of products of functions in Hardy and Lipschitz or BMO spaces can generalize in this general setting. Remark that the boundary of the pseudo-convex domain in $\mathbb{C}^n$, with the metric that is adapted to the complex geometry (see for instance [McN]), gives a fundamental example of such a space of homogeneous type. The Calderón–Zygmund theory has been developed in this context, see [KL] for instance, in relation to the properties of holomorphic functions, reproducing formulas, and Bergman and Szegö projections. Many recent contributions have been made in $\mathcal{H}^p$ theory on spaces of homogeneous type. We refer to [GLY] and the references given there. Tools developed by Macías, Segovia, and their collaborators play a fundamental role, as, for instance, in the atomic decomposition of Hardy–Orlicz spaces given by Viviani (see [V] and [BG]).

2 Prerequisites on Hardy and Lipschitz spaces

We recall here the definitions and properties that we will use later on. We follow the book of Stein [St].

Let us first recall the definition of the maximal operator used for the definition of Hardy spaces. We fix a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ having integral 1 and support in $\{|x| < 1\}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ and x in $\mathbb{R}^n$, we put

$$f * \varphi(x) := \langle f, \varphi(x - \cdot)\rangle \,, \tag{8}$$

and define the maximal function $\mathcal{M}_\varphi f$ by

$$\mathcal{M}_\varphi f(x) := \sup_{t>0} \left|\left(f * \varphi_t\right)(x)\right| , \tag{9}$$

where $\varphi_t(x) = t^{-n}\varphi\left(t^{-1}x\right)$. We also define the truncated version of the maximal function, namely

$$\mathcal{M}_\varphi^{(1)} f(x) := \sup_{0<t<1} \left|\left(f * \varphi_t\right)(x)\right| . \tag{10}$$

For $p > 0$, a tempered distribution f is said to belong to the Hardy space $\mathcal{H}^p(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{H}^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \mathcal{M}_\varphi f(x)^p dx\right)^{\frac{1}{p}} < \infty. \tag{11}$$

The localized versions of Hardy spaces are defined in the same spirit, with the truncated maximal function in place of the maximal function. Namely, a tempered distribution f is said to belong to the space $\mathfrak{h}^p(\mathbb{R}^n)$ if

$$\|f\|_{\mathfrak{h}^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \mathcal{M}_\varphi^{(1)} f(x)^p dx\right)^{\frac{1}{p}} < \infty. \tag{12}$$

Recall that, up to equivalence of corresponding norms, the space $\mathcal{H}^p(\mathbb{R}^n)$ (resp. $\mathfrak{h}^p(\mathbb{R}^n)$) does not depend on the choice of the function φ. So, in what follows, we shall use the notation $\mathcal{M}f$ instead of $\mathcal{M}_\varphi f$ (resp. $\mathcal{M}^{(1)}f$ instead of $\mathcal{M}_\varphi^{(1)} f$).

Hardy–Orlicz spaces are defined in a similar way. Given a continuous function $\mathcal{P} : [0,\infty) \to [0,\infty)$ increasing from zero to infinity (but not necessarily convex, $\mathcal{P}$ is called the Orlicz function), the Orlicz space $L^{\mathcal{P}}$ consists of measurable functions f such that

$$\|f\|_{L^{\mathcal{P}}} := \inf\left\{k > 0\,;\int_{\mathbb{R}^n} \mathcal{P}\left(k^{-1}|f|\right)\, dx \leqslant 1\right\} < \infty\,. \tag{13}$$

Then $\mathcal{H}^{\mathcal{P}}$ (resp. $\mathfrak{h}^{\mathcal{P}}$) is the space of tempered distributions f such that $\mathcal{M}f$ is in $L^{\mathcal{P}}$ (resp. $\mathcal{M}^{(1)}f$ is in $L^{\mathcal{P}}$). We will be particularly interested in the choice of the function Φ given in (6) as the Orlicz function. It is easily seen that the function Φ is equivalent to a concave function (take $t/(\log(c+t))$, for c large enough). So there is no norm on the space L^Φ. In general, $\|\cdot\|_{L^{\mathcal{P}}}$ is homogeneous, but is not sub-additive. Nevertheless (see [BIJZ]),

$$\|f+g\|_{L^\Phi} \leq 4\left(\|f\|_{L^\Phi} + \|g\|_{L^\Phi}\right). \tag{14}$$

Definition 2.1 *L^Φ_* is the space of functions f such that*

$$\|f\|_{L^\Phi_*} := \sum_{j\in\mathbb{Z}^n} \|f\|_{L^\Phi(j+\mathbb{Q})} < \infty,$$

where $\mathbb{Q}$ is the unit cube centered at 0.

We accordingly define $\mathcal{H}^\Phi_*$ (resp. $\mathfrak{h}^\Phi_*$). Using the concavity described above, we have $\Phi(st) \leq Cs\Phi(t)$ for $s > 1$. It follows that L^Φ is contained in L^Φ_* as a consequence of the fact that $\|f\|_{L^\Phi(j+\mathbb{Q})} \leq \int_{j+\mathbb{Q}} \Phi(|f|)dx$. The converse inclusion is not true.

We will restrict to $p \leq 1$, since otherwise Hardy spaces are just Lebesgue spaces. We will need the atomic decompositions of the spaces $\mathcal{H}^p(\mathbb{R}^n)$ (resp. $\mathfrak{h}^p(\mathbb{R}^n)$), which we recall now.

Definition 2.2 *Let $0 < p \leq 1 < q \leq \infty$, $p < q$, and let s be an integer. A (p,q,s)-atom related to the ball B is a function $a \in L^q(\mathbb{R}^n)$ which satisfies the following conditions:*

$$\operatorname{support}(a) \subset B \quad \text{and} \quad \|a\|_q \leq |B|^{\frac{1}{q}-\frac{1}{p}}, \tag{15}$$

$$\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \quad , \quad \text{for} \;\; 0 \leq |\alpha| \leq s. \tag{16}$$

Here α varies among multi-indices, x^α denotes the product $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Condition (16) is called the moment condition.

The atomic decomposition of $\mathcal{H}^p(\mathbb{R}^n)$ is as follows. Let us fix $q > p$ and $s > n\left(\frac{1}{p} - 1\right)$. Then a tempered distribution f is in $\mathcal{H}^p(\mathbb{R}^n)$ if and only if there exists a sequence of (p, q, s)-atoms a_j and constants λ_j such that

$$f := \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{and} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty, \tag{17}$$

where the first sum is assumed to converge in the sense of distributions. Moreover, f is the limit of partial sums in $\mathcal{H}^p(\mathbb{R}^n)$, and $\|f\|_{\mathcal{H}^p(\mathbb{R}^n)}$ is equivalent to the infimum, taken on all such decompositions of f, of the quantities $\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}}$.

For the local version, we consider other kinds of atoms when the balls B are large. We have the following, where we have fixed $q > p$ and $s > n\left(\frac{1}{p} - 1\right)$. A tempered distribution f is in $\mathfrak{h}^p(\mathbb{R}^n)$ if and only if there exists a sequence of functions a_j, constants λ_j, and balls B_j for which (17) holds, and such that

(i) when $|B_j| \leq 1$, then a_j is a (p, q, s)-atom related to B_j;
(ii) when $|B_j| > 1$, then a_j is supported in B_j and

$$\|a_j\|_q \leq |B_j|^{\frac{1}{q}-\frac{1}{p}}.$$

In other words, one still has the atomic decomposition, except that for large balls one does not ask for any moment condition on atoms.

Next, let us define Lipschitz spaces. For $\delta \in \mathbb{R}^n$ we note $D_\delta^1 = D_\delta$ the difference operator, defined by setting $D_\delta f(x) = f(x+\delta) - f(x)$ for f a continuous function (see [Gr], for instance). Then, by induction, we define $D_h^{k+1} f = D_\delta \left(D_\delta^k f\right)$ for k a nonnegative integer, so that

$$D_\delta^k f(x) = \sum_{s=0}^{k} (-1)^{k+s} \binom{k}{s} f(x + s\delta). \tag{18}$$

For $\gamma > 0$ and $k = \lfloor \gamma \rfloor$ the integer part of γ, we set

$$\|f\|_{\Lambda_\gamma} = \|f\|_{L^\infty} + \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{\left|D_\delta^{k+1} f(x)\right|}{|\delta|^\gamma}. \tag{19}$$

$\Lambda_\gamma(\mathbb{R}^n)$, the inhomogeneous Lipschitz space of order γ, is defined as the space of continuous functions f such that $\|f\|_{\Lambda_\gamma} < \infty$. It is well known that $f \in$

$\Lambda_\gamma(\mathbb{R}^n)$ is of class $\mathcal{C}^k(\mathbb{R}^n)$, with $k < \gamma$. Moreover, for α a multi-index with $|\alpha| \leq k$,

$$\|\partial^\alpha f\|_{\Lambda_{\gamma-|\alpha|}} \leq C(n,\gamma)\, \|f\|_{\Lambda_\gamma} . \tag{20}$$

Similarly, we define the homogeneous Lipschitz $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ with

$$\|f\|_{\dot{\Lambda}_\gamma} = \sup_{x\in\mathbb{R}^n} \sup_{\delta\in\mathbb{R}^n\setminus\{0\}} \frac{\left|D_\delta^{k+1} f(x)\right|}{|\delta|^\gamma}, \tag{21}$$

3 Proofs of Theorem 1.1 and Theorem 1.2

Proof (Proof of Theorem 1.1) To simplify the notation, we will write $\mathcal{H}^p$ in place of $\mathcal{H}^p(\mathbb{R}^n)$, BMO in place of BMO$(\mathbb{R}^n)$, etc. The proof is inspired by the one given in [BIJZ] for the product $b \times h$ when b is in BMO. Recall that we assume that $b \in \mathfrak{bmo}$. The function $h \in \mathcal{H}^1$ admits an atomic decomposition with bounded atoms,

$$h := \sum_j \lambda_j a_j \quad , \quad \sum_j |\lambda_j| \leq C\|h\|_{\mathcal{H}^1}.$$

When the sequence h_ℓ tends to h in $\mathcal{H}^1$, the product $b \times h_\ell$ tends to $b \times h$ as a distribution. So we can write

$$b \times h = \sum_j \lambda_j (b \times a_j),$$

where the limit is taken in the distribution sense. Since the a_j are bounded functions with compact support, the product $b \times a_j$ is given by the ordinary product. We want to write $b \times h := h^{(1)} + h^{(2)}$, with $h^{(1)} \in L^1$ and $h^{(2)} \in \mathcal{H}^\Phi_*$. Let us write, for each term a_j, which is assumed to be adapted to B_j,

$$b \times a_j = (b - b_{B_j})a_j + b_{B_j} a_j.$$

By the BMO property as well as the fact that $|a_j| \leq |B_j|^{-1}$, we have the inequality

$$\sum_j |\lambda_j| \int_{\mathbb{R}^n} |b - b_{B_j}|\, |a_j| dx \leq C\|b\|_{\mathfrak{bmo}}\, \|h\|_{\mathcal{H}^1}.$$

Here $\|b\|_{\mathfrak{bmo}}$ is the sum of the two finite quantities that appear in the definition of $\mathfrak{bmo}$ given by (4). We say

$$h^{(1)} := \sum_j \lambda_j (b - b_{B_j})a_j,$$

which is the sum of a normally convergent series in L^1. Since convergence in L^1 implies convergence in the distribution sense, it follows that $h^{(2)}$ is

$$h^{(2)} := \sum_j \lambda_j b_{B_j} a_j,$$

which is well defined in the distribution sense. Moreover,

$$\begin{aligned}\mathcal{M}h^{(2)} &\leq \sum_j |\lambda_j||b_{B_j}|\mathcal{M}a_j \\ &\leq \sum_j |\lambda_j||b - b_{B_j}|\mathcal{M}a_j + |b|\sum_j |\lambda_j|\mathcal{M}a_j.\end{aligned}$$

The first term is in L^1 since $\mathcal{M}a_j \leq |B_j|^{-1}$. In order to conclude, we have to prove that the second term is in L^{Φ}_*. We first use the fact that $\|\mathcal{M}a_j\|_1 \leq C$ for some uniform constant C, which is classical and may be found in [St] for instance. Then we have to prove that, for $\psi \in L^1$, the product $b\psi$ is in L^{Φ}_*. We claim that b belongs uniformly to the exponential class on each ball of measure 1. Indeed, by the John–Nirenberg inequality which is valid for b, for some constant C, which depends only on the dimension, and for each ball B such that $|B| = 1$,

$$\int_B \exp\left(\frac{|b(x) - b_B|}{C\|b\|_{\mathfrak{bmo}}}\right)\, dx \ \leqslant 2\,. \tag{22}$$

Moreover, since $b \in \mathfrak{bmo}$, we have the inequality $|b_B| \leq \|b\|_{\mathfrak{bmo}}$. To prove that $b\psi$ is in L^{Φ}_*, we consider each such ball separately. We use the following lemma, which is an adaptation of lemmas given in [BIJZ].

Lemma 3.1 *If the integral on B of* $\exp|b|$ *is bounded by* 2, *then, for some constant C,*

$$\|b\psi\|\|_{L^{\Phi}(B)} \leq C\int_B |\psi|dx.$$

Proof By homogeneity it is sufficient to find some constant c such that, for $\int_B |\psi|dx = c$, we have

$$\int_B \frac{|b\psi|}{\log(e + |b\psi|)}dx \leq 1.$$

If we cut the integral into two parts depending on the fact that $|b| < 1$ or not, we conclude directly that the first part is bounded by c, since we have a majorant by suppressing the denominator. For the second part, we can suppress b in the denominator. Then, we use the duality between the $L\log L$ class, and the exponential class. It is sufficient to prove that the Luxemburg norm of $\frac{|\psi|}{\log(e+|\psi|)}$ in the class $L\log L$ is bounded by $1/2$ for c small enough, which is elementary.

We have an estimate for each cube $j + \mathbb{Q}$. This finishes the proof of Theorem 1.1.

Since $\mathfrak{bmo}$ is the dual of $\mathfrak{h}^1$, it is natural to see what is valid for $h \in \mathfrak{h}^1$. We can state the following.

Theorem 3.2 *For h a function in $\mathfrak{h}^1(\mathbb{R}^n)$ and b a function in $\mathfrak{bmo}(\mathbb{R}^n)$, the product $b \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$b \times h \in L^1(\mathbb{R}^n) + \mathfrak{h}^{\Phi}_*(\mathbb{R}^n). \tag{23}$$

Proof Again, we start from the atomic decomposition of h. In view of (14), it is sufficient to consider only those atoms a_j that are adapted to balls B such that $|B| \geq 1$. Remember that they do not satisfy the moment condition (16). This one was only used to ensure that $\|\mathcal{M}a_j\|_1 \leq C$ for some independent constant. We now have $\|\mathcal{M}^{(1)}a_j\|_1 \leq C$ since $\mathcal{M}^{(1)}a_j$, which is bounded by $|B_j|^{-1}$, is supported in the ball of the same center as B_j and radius twice the radius of B_j. Except for this point, the proof is identical.

Before leaving the case $p = 1$, let us add some remarks. Multipliers of the space BMO have been characterized by Stegenga in [S] (see also [CL]) when $\mathbb{R}^n$ is replaced by the torus. It is easy to extend this characterization to $\mathfrak{bmo}$. Let us first define the space $\mathfrak{lmo}$ as the space of locally integrable functions b such that

$$\sup_{|B|\leq 1} \left(\frac{\log(e + 1/|B|)}{|B|} \int_B |b - b_B| dx \right) < \infty \quad \text{and} \quad \sup_{|B|\geq 1} \left(\frac{1}{|B|} \int_B |b| dx \right) < \infty. \tag{24}$$

$\mathfrak{lmo}$ stands for *logarithmic mean oscillation.*

Proposition 3.3 *Let ϕ be a locally integrable function. Then the following properties are equivalent.*

(i) *The function ϕ is bounded and belongs to the space $\mathfrak{lmo}$.*
(ii) *For every $b \in \mathfrak{bmo}$, the function $b\phi$ is in $\mathfrak{bmo}$.*

Proof We give a direct proof, which is standard, for completeness. The proof of (i)⇒(ii) is straightforward. Indeed, let us first consider balls B such that $|B| \leq 1$. Writing $b = (b - b_B) + b_B$, we conclude directly for the first term, and have to prove that

$$|b_B| \times \frac{1}{|B|} \int_B |\phi - \phi_B| dx \leq C \|b\|_{\mathfrak{bmo}} \|\phi\|_{\mathfrak{lmo}}.$$

Let B' be the ball of same center as B and radius 1. It is well known that the fact that b is in BMO implies that

$$|b_B - b_{B'}| \leq C \log(e + 1/|B|) \|b\|_{\mathrm{BMO}}.$$

We conclude, using the fact that $|b_{B'}| \leq C \|b\|_{\mathfrak{bmo}}$. The proof is even simpler for balls B such that $|B| \geq 1$.

Conversely, assume that we have (ii). Taking $b=1$, we already know that ϕ is in $\mathfrak{bmo}$. Also, by the closed graph theorem, we know that there exists some constant C such that, for every $b\in\mathfrak{bmo}$, the function

$$\|b\phi\|_{\mathfrak{bmo}}\leq C\|b\|_{\mathfrak{bmo}}.$$

We first claim that ϕ is bounded. By the Lebesgue differentiation theorem, it is sufficient to prove that, for each ball B, the mean ϕ_B is bounded. But $\phi_B=\langle\phi,|B|^{-1}\chi_B\rangle=\langle b\phi,a\rangle$, where a is some atom of $\mathfrak{h}^1$ and b is bounded by 1. Indeed, the characteristic function χ_B may be written as the square of a function of mean zero, taking values ±1 on B. So ϕ_B is bounded. Now, since ϕ is bounded, the assumption implies that, for a ball B such that $|B|\leq 1$,

$$|b_B|\times\frac{1}{|B|}\int_B|\phi-\phi_B|dx\leq C\|b\|_{\mathfrak{bmo}}.$$

It is sufficient to find a function b with norm bounded independently of B and such that $|b_B|\geq c\log(e+1/|B|)$. The function $\log(|x-x_B|^{-1})$, with x_B the center of B, has this property.

The previous proposition allows an interpretation in view of Theorem 1.1. The duals of Hardy–Orlicz spaces have been studied by S. Janson [J], see also the work of Viviani [V], where duality is deduced from their atomic decomposition. In particular, the dual of the space $\mathfrak{h}^{\Phi}$ is the space $\mathfrak{lmo}$. It follows that the dual of the space $L^1+\mathfrak{h}^{\Phi}$ is the space $L^\infty\cap\mathfrak{lmo}$. So if a duality argument was possible, which is not the case since we are not dealing with Banach spaces, we would conclude that multiplication by $\mathfrak{bmo}$ maps $\mathfrak{h}^1$ into $L^1+\mathfrak{h}^{\Phi}$. Recall that we have a weaker statement.

Proof (Proof of Theorem 1.2) When $p>\frac{n}{n+1}$, the proof is an easy adaptation of the previous one. We start again from an atomic decomposition of h and define $h^{(1)}$ and $h^{(2)}$ as before. To conclude for $h^{(1)}\in L^1$, it is sufficient to prove that, for all balls B, one has

$$\int_B|b-b_B|dx\leq C|B|^{\frac{1}{p}}\|b\|_{\Lambda_\gamma}.$$

If B has center x_B and radius r, it follows at once from the inequality $|b(x)-b(x_B)|\leq r^\gamma\|b\|_{\Lambda_\gamma}\leq|B|^{\gamma/n}\|b\|_{\Lambda_\gamma}$, and the choice $\gamma=n(1/p-1)$.

Next we conclude directly for $h^{(2)}$, using the fact that b is bounded, so that $\mathcal{M}h^{(2)}\leq\|b\|_\infty\sum_j|\lambda_j|\mathcal{M}a_j$. This last quantity is in L^p since

$$\int|\mathcal{M}h^{(2)}|^p\leq\|b\|_\infty\sum_j|\lambda_j|^p\int|\mathcal{M}a_j|^p$$

and the $\mathcal{M}a_j$'s are uniformly in L^p.

For smaller values of p, we start again from an atomic decomposition of h, but choose the atoms a_j to be (p, ∞, s) for s to be chosen later, that is, to satisfy the moment condition (16) up to order s. We then have to modify the choice of $h^{(1)}$ and $h^{(2)}$ in order to be able to treat the first term as above. We use the following definition.

Definition 3.4 *For f a locally square integrable function and B a ball in $\mathbb{R}^n$, we define $P_B^k f$ as the orthogonal projection in $L^2(B)$ of f onto the space of polynomials of degree $\leq k$.*

The next lemma is classical. It is the easy part of the identification of Lipschitz spaces with spaces of Morrey–Campanato, see [C]. We give its proof for completeness.

Lemma 3.5 *Let $\gamma > 0$ and $k > \gamma$. There exists a constant C such that, for f a function in $\Lambda_\gamma(\mathbb{R}^n)$ and B a ball in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \int_B \left| f(x) - P_B^k f(x) \right| dx \leq C \, \|f\|_{\Lambda_\gamma} |B|^{\frac{\gamma}{n}} . \tag{25}$$

Proof In fact, we prove an L^2 inequality instead of an L^1, which is better. In this case, it is sufficient to prove the same inequality with $P_B^k f$ replaced by some polynomial P of degree $\leq k$. This allows us to conclude for γ not an integer. Indeed, take for P the Taylor polynomial at point x_B (assuming that B has center x_B and radius r) and order $\lfloor \gamma \rfloor$, using the fact that it makes sense by (20). Then, by Taylor's formula, $|f - P|$ is bounded on B by $Cr^\gamma \leq C|B|^{\gamma/n}$. For γ an integer, we conclude for (25) by interpolation.

Let us come back to the proof of Theorem 1.2. We start again from an atomic decomposition of f. We fix $k \geq \gamma$ and pose

$$h^{(1)} := \sum_j \lambda_j (b - P_{B_j}^k b) a_j .$$

Using the previous lemma, we conclude as before that $h^{(1)}$ is in L^1. In order to have $h^{(2)} := b \times h - h^{(1)}$ in $\mathcal{H}^p$, it is sufficient that each term $(P_{B_j}^k b) a_j$ be, up to the multiplication by a uniform constant, a (p, ∞, s')-atom with $s' \geq \gamma$. The moment condition is clearly satisfied if we have $s \geq k + s'$. We can, in particular, choose $k = s' = \lfloor \gamma \rfloor + 1$ and $s = 2\lfloor \gamma \rfloor + 2$. It remains to prove that $P_{B_j}^k$ is uniformly bounded. This follows from the following lemma.

Lemma 3.6 *Let k be a positive integer. There exists a constant $C > 0$ such that for every ball B in $\mathbb{R}^n$,*

$$\left\| P_B^k f \right\|_{L^\infty(B)} \leq C \, \|f\|_{L^\infty(B)} , \tag{26}$$

for all functions f which are bounded on the ball B.

Proof We remark first that, by invariance by translation, we can assume that B is centered at 0. Next, by invariance by dilation, we can also assume that $|B| = 1$. So we have to prove it for just one fixed ball. Now, since the projection is done on a finite-dimensional space,

$$\left\|P_B^k f\right\|_{L^\infty(B)} \leq C_k \left\|P_B^k f\right\|_{L^2(B)} \leq C_k \left\|f\right\|_{L^2(B)} \leq C_k \left\|f\right\|_{L^\infty(B)} .$$

This allows to conclude for the proof of the theorem.

As for the case $p = 1$, we can take h in the local Hardy space.

Theorem 3.7 *For h a function in $\mathfrak{h}^p(\mathbb{R}^n)$ and b a function in $\Lambda_\gamma(\mathbb{R}^n)$, the product $b \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion*

$$b \times h \in L^1(\mathbb{R}^n) + \mathfrak{h}^p(\mathbb{R}^n). \tag{27}$$

Proof The adaptation of the previous proof is done in the same way as we have done for Theorem 3.2 compared to Theorem 1.1. We leave it to the reader.

We did not give estimates of the norms, but it follows from the proof of Theorem 1.2 that we have the inequality

$$\|h^{(1)}\|_1 + \|h^{(2)}\|_{\mathcal{H}^p} \leq C\|h\|_{\mathcal{H}^p} \times \|b\|_{\Lambda_\gamma(\mathbb{R}^n)}.$$

So the bilinear operator

$$\begin{aligned} \mathfrak{P} : \Lambda_\gamma(\mathbb{R}^n) \times \mathcal{H}^p(\mathbb{R}^n) &\to L^1(\mathbb{R}^n) + \mathcal{H}^p(\mathbb{R}^n) \\ (b, h) &\mapsto \qquad b \times h \end{aligned}$$

is continuous. It is easy to see that the term in L^1 is present in general: for instance, take an example in which the product is positive. The same remarks are valid for all three other cases.

Remark 1 The product of $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$ and $h \in \mathcal{H}^p(\mathbb{R}^n)$ is also well defined. It belongs to some $L^1(\mathbb{R}^n) + \mathcal{H}^p_w(\mathbb{R}^n)$ with a weight w conveniently chosen. Now b is no longer bounded, but can increase as $|x|^\gamma$ at infinity. We can take any weight $(1 + |x|)^{-\alpha}$, with $\alpha > \gamma p$.

4 Generalization to spaces of homogeneous type

All proofs generalize easily to spaces of homogeneous type once one has been able to define correctly the product $b \times h$. We will not give the details of the terminology and proofs when the generalization may be done without any difficulty, but will essentially concentrate on the definition of the product.

Let us first recall some definitions. We assume that we are given a locally compact Hausdorff space X, endowed with a quasi-metric d and a positive regular measure μ such that the doubling condition

$$0 < \mu\left(B_{(x,2r)}\right) \le C\mu\left(B_{(x,r)}\right) < +\infty \tag{28}$$

holds, for all x in X and $r > 0$. Here, by a quasi-metric d, we mean a function $d : X \times X \to [0; +\infty[$ which satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all x, y in X;
(iii) there exists a finite constant $\kappa \ge 1$ such that

$$d(x, y) \le \kappa\left(d(x, z) + d(z, y)\right) \tag{29}$$

for all x, y, z in X.

Given $x \in X$ and $r > 0$, we denote $B_{(x,r)} = \{y \in X : d(x, y) < r\}$ the ball with center x and radius r.

Definition 4.1 *We call the space of homogeneous type (X, d, μ) such a locally compact space X, given together with the quasi-metric d and the nonnegative Borel measure μ on X that satisfies the doubling condition.*

On such a space of homogeneous type we can define the BMO(X) and $\mathfrak{bmo}(X)$ spaces and have the John–Nirenberg inequality. Just replace Euclidean balls by the balls on X, and the Lebesgue measure dx by the measure $d\mu$. The Hardy space $\mathcal{H}^1(X)$ can be defined by the atomic decomposition and we have the duality $\mathcal{H}^1$-BMO. But we want also to define the product $b \times h$ for $b \in \mathfrak{bmo}(X)$ and $h \in \mathcal{H}^1(X)$, while we can no longer speak of distributions. In view of Theorem 3.3, which generalizes easily in this context, we can take $\mathcal{C} \cap L^\infty \cap \mathfrak{lmo}$ as a space of test functions, but we need to have density theorems of such functions in the space of continuous compactly supported functions to recover the pointwise product when it is integrable. We encounter this difficulty a fortiori when dealing with Lipschitz spaces.

Macías and Segovia have overcome this kind of difficulty for being able to develop the theory of $\mathcal{H}^p$ spaces for $p < 1$ (see also [U]). We will assume that the measure μ does not charge points for simplification. They have proved [MS1] that, without loss of generality, X may be assumed to be a *normal space of homogeneous type and of order* $\alpha > 0$ when, eventually, the quasi-distance is replaced by an equivalent one. That is, the quasi-metric d and the measure μ are assumed to satisfy the following properties.

There exist four positive constants A_1, A_2, K_1, and K_2, such that

$$A_1 r \le \mu(B_{(x,r)}) \le A_2 r \text{ if } 0 \le r \le K_1\mu(X) \tag{30}$$

$$B_{(x,r)} = X \text{ if } r > K_1\mu(X) \tag{31}$$

$$|d(x, z) - d(y, z)| \le K_3 r^{1-\alpha} d(x, y)^\alpha, \tag{32}$$

for every x, y, and z in X, whenever $d(x, z) < r$ and $d(y, z) < r$.

In the Euclidean case, a normal quasi-distance is given by $|x - y|^n$. The assumption (32) is satisfied with $\alpha = 1/n$.

Let us then define the Lipschitz spaces, as is natural, by the following.

Definition 4.2 *Let $\gamma > 0$. The Lipschitz space $\Lambda_\gamma(X, d, \mu)$ consists of bounded continuous functions f on X for which, for some constant C and all x, y,*

$$|f(x) - f(y)| \leq Cd(x, y)^\gamma. \tag{33}$$

Note that there is a change of parameter in the Euclidean space; γ has been changed into γ/n.

Remark that with these definitions Lipschitz spaces are contained in $\mathfrak{bmo}$.

We know that the space is not reduced to 0 when γ is not larger than α because of the fact that the distance itself satisfies this kind of condition. Macías and Segovia have proved in [MS2] that one can build approximate identities in order to approach continuous functions with compact support by Lipschitz functions of order γ, for any $\gamma < \alpha$. Moreover, they define the space of distributions $(E^\alpha)^*$ as the dual of the space E^α, consisting of all functions with bounded support, belonging to Λ_β for every $0 < \beta < \alpha$. From this point, they can use distributions to define $\mathcal{H}^p$ spaces when $p > (1+\alpha)^{-1}$. We recover in the Euclidean case the condition $p > n/(n+1)$, which is the range where atoms are assumed to satisfy only the moment condition of order zero, and where the dual is a Lipschitz space defined by a condition implying only one difference operator.

This notion of distribution is exactly what we need for the definition of products. With the conditions above and in the corresponding range of p, products may be defined in the distribution sense, and the four theorems are valid.

Remark that, for X the boundary of a bounded smooth pseudo-convex domain of finite type, Lipschitz spaces can be defined for all values of γ, and $\mathcal{H}^p$ spaces can be defined for p arbitrarily small. The moment conditions of higher order rely on the use of vector fields related to the geometric structure of the boundary. We refer to [McN] for the geometrical aspects, and to [BG] for the detailed statements related to products of holomorphic functions in $\mathcal{H}^p$ and BMO or Lipschitz spaces.

Acknowledgements

Part of this work was done while the second author was visiting MAPMO at Orléans, with the financial support of AUF (Agence Universitaire de la Francophonie).

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Harmonic Analysis Related to Hermite Expansions

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Summary. In this chapter we give the state of the art of harmonic analysis associated with mainly two orthogonal systems: Hermite polynomials and Hermite functions. For the sake of understanding the *global part* of several operators appearing within the context of Hermite polynomials we give in some details our contributions on the subject.

Key words: Gaussian harmonic analysis, Hermite polynomials, Hermite functions, orthogonal systems.

1 Introduction

Let us start by considering the classical d-dimensional heat semigroup, $T_t = e^{t\Delta}$. As it is well known, when applied to a good enough function, it provides a solution to the heat-diffusion equation with initial data f, that is, $u(x,t) = T_t f(x)$ solves the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & t > 0, \\ u(x,0) = f(x). \end{cases}$$

In this case, the semigroup is given by the convolution with a nice kernel, the Gauss–Weierstrass kernel, namely

$$T_t = \mathcal{W}_t * f$$

with

$$\mathcal{W}(x) = \frac{1}{(4\pi)^{d/2}} e^{-\frac{|x|^2}{4}}$$

and $\mathcal{W}_t(x) = t^{-d/2}\mathcal{W}(\frac{x}{t^{1/2}})$.

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_5,

Since such a convolution makes sense (and gives a smooth function), even if f just belongs to $L^p(dx)$, where dx is the Lebesgue measure, an important question is to determine whether and in what sense $u(.,t) \to f$ when $t \to 0$. In particular, it is of great interest to know if such a limit occurs in the almost everywhere sense. The classical approach to that problem involves the study of L^p-boundedness properties of the associated maximal operator

$$T^* f = \sup_{t>0} |T_t f|.$$

As is well known, such an operator is bounded on $L^p(dx)$ for $1 < p \le \infty$ and of weak type for $p = 1$. This is a consequence of some easy-to-check properties of the kernel: it is non-negative, radial, decreasing, and integrable. Then, by a general result on approximations to the identity, [34], we have $T^* f \le Mf$, where M denotes the Hardy–Littlewood maximal function.

Let us isolate some features of the Laplace operator Δ or rather $-\Delta$: it is a second order elliptic differential operator, non-negative and self-adjoint on $C_o^\infty(\mathcal{R}^d)$, the dense subspace of $L^2(dx)$ of C^∞-functions with compact support, that is

$$\int_{\mathcal{R}^d} \Delta\Phi \; \Psi \; dx = \int_{\mathcal{R}^d} \Phi \; \Delta\Psi \; dx,$$

for any $\Phi, \Psi \in C_o^\infty(\mathcal{R}^d)$.

Now, we pose a more general case of a second order differential operator L, acting on functions defined on X, with X a subset of $\mathcal{R}^d$, such that L is non-negative and self-adjoint with respect to some absolutely continuous measure $d\mu = \mu(x)dx$, that is,

$$\int_X L\Phi \; \Psi \; d\mu = \int_X \Phi \; L\Psi \; d\mu,$$

for Φ, Ψ in $C_o^\infty(X)$.

In this context we may consider the diffusion semigroup generated by L, $T_t = e^{-tL}$, and ask ourselves the same questions as above, where now we hope to solve the *diffusion* equation

$$\begin{cases} \frac{\partial u}{\partial t} = -Lu, & \text{for } t > 0, \; x \in X \\ u(x,0) = f(x), & x \in X. \end{cases}$$

In particular, we may ask ourselves about the boundedness properties of T_t and T^* on the spaces $L^p(X, d\mu)$ in order to get the almost everywhere convergence of the solution to some initial data f belonging to $L^p(X, d\mu)$. At this point it is worth mentioning that in many cases the main interest relies on $p = 1$, since for $p > 1$ this result follows from a general semigroup theory developed in Stein's book [35].

As in the classical case, we may also consider the Poisson semigroup $\{P_t\}_{t>0}$, whose infinitesimal generator is given by $L^{1/2}$. When applied to some appropriate function f, it should provide a solution to the Laplace-type equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = Lu, & t > 0, x \in X \\ \\ u(x,0) = f(x), & x \in X. \end{cases}$$

Here again, the question on the almost everywhere convergence of the solution to the initial data $f \in L^p(X, d\mu)$ arises. From the real variable identity

$$e^{-t\sqrt{\gamma}} = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-\frac{t^2}{4s}} e^{-s\gamma} s^{-3/2} ds, \ \gamma > 0,$$

we may expect the following subordination formula between P_t and T_t:

$$P_t f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-\frac{t^2}{4s}} T_s f(x) s^{-3/2} ds,$$

leading to the inequality

$$P^* f(x) \le T^* f(x),$$

where P^* denotes the maximal operator associated with the Poisson semigroup.

Along this line, L-Riesz potentials, i.e., negative powers of the operator L, could also be defined by

$$L^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty T_t f t^{\sigma-1} dt, \ \sigma > 0$$

in view of the identity $s^{-\sigma} = \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^\infty e^{-ts} t^{\sigma-1} dt, \ \sigma > 0.$

We remark that this formula is not exactly right when the operator L has 0 as an eigenvalue: we might have to modify T_t by subtracting a constant times de identity, in order to avoid a singularity in the above integral. Nevertheless, along this introduction we will keep the same notation for those potentials as well.

Other natural and useful operators in classical harmonic analysis are Riesz transforms and Littlewood–Paley–Stein g functions.

Before discussing their definition, we are going to assume further properties on L. Unlike the Laplace operator, we are going to focus on operators L having a discrete set of eigenfunctions $\{\phi_k\}_{k=1}^\infty$, with eigenvalues, say λ_k, and such that they form a complete orthonormal set on $L^2(X, d\mu)$. From our previous assumptions we know that $\lambda_k \ge 0$, and we list them as an increasing sequence going to infinity. As examples of such situations, we can mention some of the classical *orthogonal expansions*, like the ones with Hermite and Laguerre polynomials, and Hermite and Laguerre functions, among others. Here we illustrate how they fit in this setting.

1. d-dimensional Hermite polynomials,

$$\mathcal{L} = -\Delta + 2x \cdot \nabla, \ d\mu = e^{-|x|^2} dx, \ X = \mathcal{R}^d.$$

2. d-dimensional Hermite functions,

$$L_d = -\Delta + |x|^2, \ d\mu = dx, \ X = \mathcal{R}^d.$$

3. 1-dimensional Laguerre polynomials, $\alpha > -1$,

$$\mathcal{L}_\alpha = -x\frac{d^2}{dx^2} - (\alpha + 1 - x)\frac{d}{dx}, \ d\mu = x^\alpha e^{-x} dx, \ X = (0, \infty).$$

4. 1-dimensional Laguerre functions, $\alpha > -1$,

$$L_\alpha = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} - \frac{\alpha^2}{4x}, \ d\mu = dx, \ X = (0, \infty).$$

Let us observe that while $\mathcal{L}$ and $\mathcal{L}_\alpha$ have $\lambda = 0$ as an eigenvalue, L_d and L_α have all positive eigenvalues.

Our intention is to deal, mainly, with the two first cases: Hermite polynomials and functions. They are also referred as the Ornstein–Uhlenbeck differential operator and the harmonic oscillator, respectively.

Carlos Segovia became interested in these topics during one of his frequent visits to the Universidad Autónoma de Madrid. There, several local mathematicians were working in the field, like F. Soria with his student S. Pérez, T. Menárguez, J. García-Cuerva, J.L. Torrea together with other foreign researchers like P. Sjögren and G.C. Mauceri, among others.

Segovia's contributions to this field include works in both Hermite and Laguerre contexts. In particular, several articles in the latter context were done in collaboration with his lifetime friend R. Macías. However, as we said, we will focus this survey on Hermite polynomials and functions.

Let us go back to our second order differential operator, non-negative and self-adjoint with respect to $d\mu$, with a discrete set of eigenvalues $\{\lambda_k\}_{k=1}^\infty$ corresponding to eigenfunctions $\{\phi_k\}_{k=1}^\infty$ which form an orthonormal basis on $L^2(X, d\mu)$. In such a case, the semigroups e^{-tL} and $e^{-t\sqrt{L}}$ are well defined on $L^2(X, d\mu)$, and also the subordination formula holds. In fact,

$$T_t f = e^{-tL} f = \sum_{k=1}^\infty e^{-t\lambda_k} \langle f, \phi_k \rangle \phi_k$$

and

$$P_t f = e^{-t\sqrt{L}} f = \sum_{k=1}^\infty e^{-t\sqrt{\lambda_k}} \langle f, \phi_k \rangle \phi_k.$$

Plancherel's formula implies that both semigroups are contractions on $L^2(X, d\mu)$. Moreover, if we assume that $\sum_{k=1}^\infty e^{-2t\lambda_k} < \infty$, replacing in the first expression

$$\langle f, \phi_k \rangle = \int_X f(y)\phi_k\ (y)\mu\ (y)dy,$$

we get that T_t can be written as an integral operator, namely

$$T_t f(x) = \int_X W(t,x,y)f(y)\mu(y)dy,$$

where $W(t,x,y) = \sum_{k=1}^{\infty} e^{-t\lambda_k}\phi_k\ (x)\phi_k(y)$. This is clear at least for functions that are finite linear combinations of ϕ_k because of the assumption on the summability of the sequence $\{e^{-2t\lambda_k}\}$.

Also, from the subordination formula, we may find a kernel for the Poisson semigroup. In fact,

$$P_t f(x) = \int_X P(t,x,y)f(y)\mu(y)dy,$$

where

$$P(t,x,y) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} te^{-t^2/2s}W(s,x,y)s^{-3/2}ds.$$

Analogously, L-Riesz potentials, $L^{-\sigma}$, may be expressed as integral operators with kernels

$$G_\sigma(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^{\infty} W(t,x,y)t^{\sigma-1}dt.$$

As a conclusion we can say that, once we have a kernel for the diffusion semigroup, through the previous formulae we can write kernels for the other operators as well. This way of writing the operators clearly holds for finite linear combinations of ϕ_k, but either in the Hermite or the Laguerre setting, they are dense on $L^p(d\mu)$, $1 \le p < \infty$ (in Laguerre's case that is true when the parameter α is non-negative).

Now, let us turn to the definitions of Riesz transforms and Littlewood–Paley–Stein g functions.

In the classical case of $L = -\Delta$, the Riesz transforms are defined for $j = 1, \ldots, d$ as

$$R_j f = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2} f,$$

and for $f \in L^p(dx)$ they can be expressed as singular integral operators with kernels resulting from taking derivatives inside the integral in t defining the kernel of $(-\Delta)^{-1/2}$, namely

$$K_j(x-y) = \frac{1}{\Gamma(1/2)}\int_0^{\infty} \frac{\partial}{\partial x_j}\mathcal{W}_t(x-y)t^{-1/2}dt,$$

where, as above, $\mathcal{W}_t$ is the Gauss–Weierstrass heat kernel. Let us notice that there is a new ingredient in this definition: the appearance of the operators

$\frac{\partial}{\partial x_j}$. Now if we consider another operator L, the key question is whether we should use the same *derivatives* to define Riesz transforms or whether they should be replaced by appropriate first order differential operators, more related to L.

A very useful formula involving $-\Delta$ and partial derivatives is

$$\int_{\mathcal{R}^d} (-\Delta\phi)(x)\psi(x)dx = \int_{\mathcal{R}^d} \nabla\phi\ (x) \cdot \nabla\psi(x)dx,$$

with $\phi, \psi \in C_o^\infty(\mathcal{R}^d)$. The above identity may be seen as a consequence of the factorization

$$-\Delta = -\nabla \cdot \nabla,$$

each factor being the adjoint of the other with respect to the Lebesgue measure dx.

Also, such a factorization is the clue for a quite important identity involving the Riesz transforms: they decompose the identity. That is, if $\mathcal{R}$ denotes the vector whose components are the Riesz transforms R_j, we have

$$I = (-\mathcal{R}) \cdot \mathcal{R},$$

$-\mathcal{R}$ being in this case the adjoint operator of $\mathcal{R}$ in the componentwise sense.

Coming back to our context, to define appropriate Riesz transforms, we should be able to factor out our operator L as

$$L = D_L^* \cdot D_L,$$

where D_L is a first order differential operator and D_L^* denotes its adjoint with respect to $d\mu$.

In the Ornstein–Uhlenbeck case, we have such a factorization, namely

$$\mathcal{L} = D_{\mathcal{L}}^* \cdot D_{\mathcal{L}}$$

with $D_{\mathcal{L}} = \nabla$ and $D_{\mathcal{L}}^* = -\nabla + 2x = -e^{|x|^2}\nabla(e^{-|x|^2})$.

For the harmonic oscillator, as Thangavelu pointed out (see [40]), the operator L_d admits the following decomposition:

$$L_d = \frac{1}{2}(A^* \cdot A + A \cdot A^*)$$

with $A_i = -\frac{\partial}{\partial x_i} + x_i$ and $A_i^* = \frac{\partial}{\partial x_i} + x_i$, $i = 1, \ldots, d$.

Let us observe that, while in the classical context, both factors in the decomposition of $-\Delta$ are basically the same, that is no longer true in the above cases; they do not even commute. In this way we are in the presence of two possible Riesz transforms: $(D_{\mathcal{L}})_i\mathcal{L}^{-1/2}$ and $(D_{\mathcal{L}}^*)_i\mathcal{L}^{-1/2}$ for the first case and $A_iL_d^{-1/2}$ and $A_i^*L_d^{-1/2}$ for the second one. Since these L-*derivatives* do not commute with $L^{-1/2}$, one of the Riesz transforms is not the adjoint operator of the other; therefore, the study of one of them cannot be reduced to the study of the other.

As a generalization of the Riesz transforms of first order defined above, we have the higher order Riesz transforms: for m a natural number, we define the mth Riesz transform to be

$$R_{j_1,j_2,\ldots,j_m} f(x) = \partial_{j_1}\partial_{j_2}\cdots\partial_{j_m} L^{-\frac{m}{2}} f\ (x) \tag{1}$$

with ∂_j being either $(D_L)_j$ or $(D_L^*)_j$.

Let us notice that once we determine the right derivatives, say D_L, we have the candidates for the Riesz transforms that we expect to be *singular integrals* with kernels

$$K_j(x,y) = \frac{1}{\Gamma(1/2)} \int_0^\infty (D_L)_j W(t,x,y) t^{-1/2} dt$$

and also the candidates associated with $(D_L^*)_j$.

As for the Littewood–Paley g functions of order one they are defined as

$$g_t f(x) = \left(\int_0^\infty |\frac{\partial}{\partial t} P_t f(x)|^2 t dt \right)^{1/2}$$

and

$$g_{x_i} f(x) = \left(\int_0^\infty |\frac{\partial}{\partial x_i} P_t f(x)|^2 t dt \right)^{1/2}$$

with P_t, the classical Poisson integral.

Therefore, we may define g functions related to an operator L replacing $\frac{\partial}{\partial x_i}$ by the appropriate derivatives. Again, since the Poisson semigroup can be expressed in terms of a kernel based on the knowledge of the heat kernel $W(t,x,y)$, the Littlewood–Paley–Stein g functions in the L-context can also be written in terms of kernels (with values on a Banach space) depending on the diffusion semigroup's kernel.

At this point we would like to remark that all the operators introduced here also have a spectral representation. Our intention in this introduction has been to center the material in the expected integral representations, and to propose a way to find the corresponding kernels in terms of $W(t,x,y)$. The question of whether the integral versions coincide with the spectral definitions remains an issue to be treated in each particular case.

Therefore, the central question that is left to answer is: can we have a friendly expression for the heat kernel $W(t,x,y)$ in order to prove the needed properties? As we said before, the kernel for the diffusion semigroup is given by

$$W(t,x,y) = \sum_{k=1}^\infty e^{-t\lambda_k} \phi_k\ (x)\phi_k(y).$$

In the Hermite context (by considering both polynomials and functions), thanks to Mehler's formula [20], the series on the right-hand side can be summed up to give

$$W_{\mathcal{L}}(t,x,y) = (\pi(1-e^{-4t})^{-d/2} e^{|x|^2} e^{-\frac{|e^{-2t}y-x|^2}{1-e^{-4t}}} \tag{2}$$

and

$$W_{L_d}(t,x,y) = (\pi(1-r^2)^{-d/2} e^{-\{\frac{r}{1+r}|x+y|^2+\frac{1+r}{1-r}|x-y|^2\}}, r = e^{-t}. \tag{3}$$

Similar summation formulas [3] can be used to obtain handy expressions in the Laguerre context. For instance, in the case of Laguerre functions, for $d = 1$, one is led to

$$W_{L_\alpha}(t,x,y) = \frac{r^{1/2}}{1-r} e^{-\frac{1}{2}\{\frac{r}{1+r}(x+y)\}} I_\alpha\left(\frac{2(rxy)^{1/2}}{1-r}\right)$$

with $r = e^{-t}$ and $I_\alpha(s) = (-i)^\alpha J_\alpha(is)$, J_α being the Bessel function, [36].

Such expressions are hence the starting point to study the whole list of operators we have described.

The chapter is organized as follows. Section 2 will be devoted to Hermite polynomials, while in Section 3 we shall review some results concerning with the harmonic oscillator operator. In each context we shall describe what is known about the following operators: the maximal heat-diffusion semigroup, the Riesz transforms, and the Littlewood–Paley–Stein quadratic functions.

Many authors have contributed to shed light on many of the problems related to the boundedness of the above operators in the setting of the semigroups associated with Hermite expansions. In the following sections we will discuss in more details the state of the art on each setting (Hermite polynomials and functions), describing Carlos Segovia's works on this subject as well as contributions from several other mathematicians.

2 Hermite polynomials

Let us recall that the Hermite polynomials are defined on $\mathcal{R}$ as

$$H_0(x) = 1, \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n \geq 1;$$

and the multidimensional Hermite polynomials are defined by taking products of one-dimensional Hermite polynomials. Indeed, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, then we define

$$H_\alpha(x) = H_{\alpha_1}(x_1) \ldots H_{\alpha_d}(x_d).$$

They are orthogonal with respect to the Gaussian measure

$$d\gamma(x) = e^{-|x|^2} dx \,.$$

When properly normalized, they form an orthonormal basis of $L^2(d\gamma)$.

As we already mentioned, they are eigenfunctions of the operator $\mathcal{L}$ and, in fact, we have

$$\mathcal{L}H_\alpha = 2|\alpha| \; H_\alpha.$$

Although the finite linear combinations of Hermite polynomials are dense on $L^p(d\gamma)$ for $1 \le p < \infty$, partial sums do not converge for functions on $L^p(d\gamma)$ unless $p = 2$, see [31].

Along this section the notation $\| \; \|_p$ stands for the L^p-norm taken with respect to the Gaussian measure.

2.1 The Ornstein–Uhlenbeck maximal operator

By making the change of parameters $r = e^{-2t}$ in (22) we get that the maximal operator corresponding to the Hermite polynomial diffusion semigroup is

$$W^* f(x) = \sup_{0<r<1} \frac{e^{|x|^2}}{(\pi(1-r^2))^{d/2}} \left| \int_{\mathcal{R}^d} e^{-\frac{|ry-x|^2}{1-r^2}} f(y) d\gamma(y) \right| . \tag{4}$$

Since $(e^{-t\mathcal{L}})_{t>0}$ is a symmetric diffusion semigroup in the sense of Stein [35], it is possible to apply the general Littlewood–Paley theory and prove its boundedness on $L^p(d\gamma)$ for $1 < p < \infty$ with a constant uniform on dimension.

On the other hand, the $L^\infty(d\gamma)$ boundedness is immediate, and if the weak type $(1,1)$ with respect to the Gaussian measure were also true, it would be possible to use interpolation theory to get another proof of the $L^p(d\gamma)$ result.

Indeed, the weak type $(1,1)$ of this maximal operator holds true and has a long history that started with two independent papers due to Calderón [4] and Muckenhoupt [24] in 1969. They proved the result for dimension $d = 1$. Both proofs consist of bounding W^* by the non-centered Hardy–Littlewood maximal functions with respect to the Gaussian measure. Due to the fact of being in one dimension, a geometric covering lemma of Young can be used in order to get the weak type $(1,1)$ inequality for such maximal functions (see [14]). In fact, this result holds for any Hardy–Littlewood maximal function associated with an absolutely continuous measure, see [8].

In higher dimensions the same estimate can be obtained, but the non-centered Hardy–Littlewood maximal function with respect to the Gaussian measure is no longer weak type $(1,1)$ [33]. Therefore, further technical work is necessary for this case.

The weak type $(1,1)$ for $d > 1$ was finally proved by Sjögren in 1983 in an impressive technical paper [32]. The method of the proof consists in splitting the integral defining W^* into a local and a global part, that had its background in a paper due to Muckenhoupt [25].

In this context, local means the hyperbolic Gaussian ball centered at x: $B_x := \{y : |x-y| < \frac{C}{1+|x|}\}$. Since the Gaussian density $e^{-|y|^2}$ is equivalent at each point of a Gaussian ball, then in (4), when the integral is restricted to the local part, $e^{-|y|^2}$ cancels out with $e^{|x|^2}$, and with a little work, it is not hard to see that this part can be controlled by the maximal function associated with the standard heat semigroup restricted to the local zone. But, any operator with a positive kernel which is weak type with respect to the Lebesgue measure turns out to be weak type with respect to the Gaussian measure once restricted to the local zone, see [28]. This takes care of the local part of the operator.

In the global region, Sjögren [32] proved that the following greater integral operator:

$$\begin{aligned} Kf(x) &= \int_{\mathcal{R}^d} K(x,y)\ |f(y)|\ d\gamma(y) \\ &= \int_{|x-y|>\frac{C}{|x|+1}} \sup_{0<r<1} \frac{e^{|x|^2}}{(\pi(1-r^2))^{d/2}} e^{-\frac{|ry-x|^2}{1-r^2}}\ |f(y)|\ d\gamma(y) \end{aligned}$$

is weak type $(1,1)$ with respect to the Gaussian measure with a method of his own that later was known as the forbidden regions technique. Roughly, he bounds the measure of the set

$$A = \{x \in \mathcal{R}^d :\ Kf(x) > \lambda\}$$

by a constant times the measure of a subset E of A in such a way that

$$\int_E K(x,y)\ d\gamma(x) \leq C.$$

Actually, this is the hardest job. From here the proof follows easily by using the fact that $E \subset A$ and using Chebyshev's inequality. In fact, we have

$$\begin{aligned} \gamma(A) &\leq C\gamma(E) \\ &\leq \frac{C}{\lambda} \int_E Kf(x)\ d\gamma(x) \\ &\leq \frac{C}{\lambda} \int_E \int_{\mathcal{R}^d} K(x,y)\ |f(y)|\ d\gamma(y)\ d\gamma(x) \\ &= \frac{C}{\lambda} \int_{\mathcal{R}^d} \int_E K(x,y)\ d\gamma(x)\ |f(y)|\ d\gamma(y) \\ &\leq \frac{C}{\lambda} \int_{\mathcal{R}^d} |f(y)|\ d\gamma(y). \end{aligned}$$

Since Sjögren's proof was ad hoc, highly technical, and difficult to generalize to other operators in the Gaussian context, there were many attempts to give simpler and more understandable proofs. Pérez, in her doctoral dissertation [26] (see also [22], [29], [28]), got an explicit bound for the kernel

$$\sup_{0<r<1} \frac{1}{(\pi(1-r^2))^{d/2}} e^{-\frac{|ry-x|^2}{1-r^2}}$$

in the global region.

In order to get that bound she computes the derivative of the kernel and gets approximately the point where the maximum is attained. The computation is not easy. It takes a lot of effort, but the interesting part of it is that such a kernel bounds the global part of several other operators, as we shall see later. Calling the latter kernel $K^*(x,y)$, she proves that the operator defined by

$$\mathcal{K}^* f(x) = Ce^{|x|^2} \int_{|y-x|\geq \frac{C}{1+|x|}} K^*(x,y)|f(y)|d\gamma(y) \tag{5}$$

is of weak type $(1,1)$.

Here we give a rewritten version of the kernel introduced by Pérez [26], [29]. We will sketch a proof of the weak type $(1,1)$ for the operator defined in (5), simpler than the original one due to Pérez and Soria [26], [29]. It makes use of a clever technique first introduced by García-Cuerva, Mauceri, Meda, Sjögren and Torrea in [12], where they prove the weak type $(1,1)$ inequality for the holomorphic extension of the operator (4). Let us remark that the kernel they consider is strictly smaller than K^* defined below. In particular, it does not provide a bound for the second order Riesz transforms as K^* does, see [7].

The kernel $K^*(x,y)$, upon a multiplicative constant, is defined over $|y-x| \geq \frac{C}{1+|x|}$ as

$$K^*(x,y) = \begin{cases} e^{-|x|^2} & \text{for } x\cdot y<0 \\ |x|^d e^{-C|x||x-y|} & \text{for } x\cdot y\geq 0 \text{ and } |x|>|y| \\ |x|^d \left(1 \wedge \dfrac{e^{-\frac{c|x|^4\alpha^2(x,y)}{|y|^2-|x|^2+\alpha(x,y)|x|^2}}}{(|y|^2-|x|^2+\alpha(x,y)|x|^2)^{\frac{d-1}{2}}}\right) & \text{for } x\cdot y\geq 0 \text{ and } |x|\leq |y|\leq 2|x| \\ (1+|x|)e^{-\alpha^2(x,y)|x|^2} & \text{for } x\cdot y\geq 0 \text{ and } |y|>2|x| \end{cases} \tag{6}$$

where $\alpha(x,y)$ represents the sine of the angle between x and y.

As it is shown in [29], the kernels $e^{-|x|^2}$ when $x\cdot y<0$ and $|x|^d e^{-c|x||x-y|}$ when $x\cdot y\geq 0$ and $|x|>|y|$ define strong type $(1,1)$ operators.

From the above remarks, to get the weak type $(1,1)$ inequality for the operator (5), it suffices to prove that the operators

$$S_0 f(x) = |x|^d e^{|x|^2} \int_{R_0} \left(1 \wedge \frac{e^{-\frac{c|x|^4 \alpha^2(x,y)}{|y|^2-|x|^2+\alpha(x,y)|x|^2}}}{(|y|^2-|x|^2+\alpha(x,y)|x|^2)^{\frac{d-1}{2}}} \right) f(y) d\gamma(y)$$

and

$$S_1 f(x) = |x| e^{|x|^2} \int_{R_1} e^{-c\alpha^2(x,y)|x|^2} f(y) d\gamma(y)$$

map $L^1(d\gamma)$ into $L^{1,\infty}(d\gamma)$. Here

$$R_0 = \{y \notin B(x, \frac{C}{1+|x|}) \text{ such that } x \cdot y \geq 0 \text{ and } |y| \in [|x|, 2|x|]\},$$

$$R_1 = \{y \notin B(x, \frac{C}{1+|x|}) \text{ such that } x \cdot y \geq 0 \text{ and } |y| > 2|x|\}.$$

Without loss of generality, we may assume $f \geq 0$. For $\lambda > 0$ let E_i be the level set $\{x \in \mathcal{R}^d : \ S_i f(x) > \lambda\}$, for $i = 0, 1$. We shall prove that $\gamma(E_i) \leq \frac{C}{\lambda}||f||_1$. Let r_0 and r_1 be the smallest positive roots of the equations

$$r_0^d e^{r_0^2} ||f||_1 = \lambda \quad \text{and} \quad r_1 e^{r_1^2} ||f||_1 = \lambda.$$

Therefore, $E_i \cap \{x \in \mathcal{R}^d : |x| < r_i\} = \emptyset$. On the other hand, since we are working on a space of finite measure, it suffices to take $\lambda > K||f||_1$, and by choosing K large enough, we may assume that both r_0 and r_1 are larger than one. Hence, $\gamma\{x \in \mathcal{R}^d : |x| > 2r_i\} \leq C r_i^{d-2} e^{-4r_i^2} \leq \frac{C}{\lambda}||f||_1$.

Thus, we only need to estimate $\gamma\{x \in E_i : r_i \leq |x| \leq 2r_i\}$.

Let $E_i' = \{x' \in S^{d-1} : \exists \rho \in [r_i, 2r_i] \text{ with } \rho x' \in E_i\}$ and for $x' \in E_i'$ let $\rho_i(x')$ be the smallest such ρ. Then $S_i f(\rho_i(x')x') = \lambda$ by the continuity of $S_i f(x)$. This implies for $i = 0$ and $x' \in E_0'$,

$$C \ e^{\rho_0^2(x')} r_0^d \int_{|y| \geq r_0} \left(1 \wedge \frac{e^{-\frac{c r_0^4 \alpha^2(x',y)}{|y|^2 - r_0^2 + \alpha(x',y) r_0^2}}}{(|y|^2 - r_0^2 + \alpha(x',y) r_0^2)^{\frac{d-1}{2}}} \right) f(y) d\gamma(y) \geq \lambda, \quad (7)$$

and for $i = 1$, $x' \in E_1'$,

$$C \ e^{\rho_1^2(x')} r_1 \int_{|y| \geq r_1} e^{-c\alpha^2(x',y) r_1^2} f(y) d\gamma(y) \geq \lambda. \quad (8)$$

Clearly, since r_0 and r_1 are greater than one, we have

$$\begin{aligned}\gamma\{x \in E_i : r_i \leq |x| \leq 2r_i\} &\leq \int_{E_i'} d\sigma(x') \int_{\rho_i(x')}^{2r_i} e^{-\rho^2} \rho^{d-1} d\rho \\ &\leq C \int_{E_i'} e^{-\rho_i^2(x')} r_i^{d-2} d\sigma(x').\end{aligned}$$

Combining this estimate for $i = 0$ with (7), we get

$$\gamma\{x \in E_0 : r_0 \leq |x| \leq 2r_0\} \leq \frac{C}{\lambda} \int_{E_0'} r_0^{2d-2} d\sigma(x')(I_0 + II_0), \tag{9}$$

with

$$I_0 = \int_{\{|y|\geq r_0,\ \alpha(x',y)r_0^2 \geq c\}} \frac{e^{-\frac{cr_0^4\alpha^2(x',y)}{|y|^2-r_0^2+\alpha(x',y)r_0^2}}}{(|y|^2 - r_0^2 + \alpha(x',y)r_0^2)^{\frac{d-1}{2}}} f(y)d\gamma(y)$$

and

$$II_0 = \int_{\{\alpha(x',y)r_0^2 \leq c\}} f(y)d\gamma(y).$$

Similarly, for $i = 1$ with (8), we obtain

$$\gamma\{x \in E_1 : r_1 \leq |x| \leq 2r_1\} \leq \frac{C}{\lambda} \int_{E_1'} r_1^{d-1} d\sigma(x')(I_1 + II_1), \tag{10}$$

with

$$I_1 = \int_{\{|y|\geq r_1,\ \alpha(x',y)r_1 \geq c\}} e^{-c\alpha^2(x',y)r_1^2} f(y)d\gamma\ (y)$$

and

$$II_1 = \int_{\{\alpha(x',y)r_1 \leq c\}} f(y)d\gamma(y).$$

One can immediately verify that

$$r_0^{2d-2} \int_{\{\alpha(x',y)r_0^2 \leq c\}} d\sigma(x') \leq C \text{ and } r_1^{d-1} \int_{\{\alpha(x',y)r_1 \leq c\}} d\sigma(x') \leq C,$$

which give, after changing the order of integration in (9) and (10), the desired estimate for the terms involving II_0 and II_1, respectively.

Now let us prove that for $|y| \geq r_0$

$$r_0^{2d-2} \int_{\{\alpha(x',y)r_0^2 \geq c\}} \frac{e^{-\frac{cr_0^4\alpha^2(x',y)}{|y|^2-r_0^2+\alpha(x',y)r_0^2}}}{(|y|^2 - r_0^2 + \alpha(x',y)r_0^2)^{\frac{d-1}{2}}} d\sigma(x') \leq C \tag{11}$$

and for $|y| \geq r_1$

$$r_1^{d-1} \int_{\{\alpha(x',y)r_1 \geq c\}} e^{-c\alpha^2(x',y)r_1^2} d\sigma(x') \leq C. \tag{12}$$

For any fixed y, we choose coordinates on S^{d-1} in such a way that the north pole is on the direction of y. Then the left-hand side of (11) can be written as

$$r_0^{2d-2} \int_{\{\sin\theta r_0^2 \geq c\}} e^{-\frac{cr_0^4 \sin^2\theta}{|y|^2 - r_0^2 + \sin\theta r_0^2}} \frac{\sin^{d-3}\theta}{(|y|^2 - r_0^2 + \sin\theta\ r_0^2)^{\frac{d-3}{2}}} \frac{\sin\theta\ d\theta}{|y|^2 - r_0^2 + \sin\theta\ r_0^2}.$$

The boundedness of this integral when restricted to the angles θ such that $\sin\theta \geq 1/2$ follows easily by using $|t|^{d-1}e^{-ct^2} \leq C$. For the remaining integral, we introduce the factor $\cos\theta$ in the integral, make the change of variables $\alpha = \sin\theta$, and obtain

$$r_0^{2d-2}\int_{\{\alpha r_0^2\geq c\}} e^{-\frac{cr_0^4\alpha^2}{|y|^2-r_0^2+\alpha r_0^2}} \frac{\alpha^{d-3}}{(|y|^2-r_0^2+\alpha r_0^2)^{\frac{d-3}{2}}}\frac{\alpha d\alpha}{|y|^2-r_0^2+\alpha r_0^2}, \tag{13}$$

which, by this other change of variables,

$$u=\frac{r_0^4\alpha^2}{|y|^2-r_0^2+\alpha r_0^2}$$

with

$$du = r_0^4\frac{2|y|^2-2r_0^2+\alpha r_0^2}{|y|^2-r_0^2+\alpha r_0^2}\frac{\alpha d\alpha}{|y|^2-r_0^2+\alpha r_0^2}\geq r_0^4\frac{\alpha d\alpha}{|y|^2-r_0^2+\alpha r_0^2},$$

turns out to be bounded by $\displaystyle\int_0^\infty e^{-cu}u^{\frac{d-3}{2}}du\leq C$.

To prove (12), a similar argument as the one above may be applied. In order to take care of the integral restricted to the angles θ for which $\sin\theta\geq 1/2$, we use again $|t|^{d-1}e^{-ct^2}\leq C$. For the remaining integral, the same argument applies in order to make the change of variables $\alpha=\sin\theta$, and therefore, in this case we obtain

$$r_1^{d-1}\int_{\{\alpha r_1\geq c\}} e^{-cr_1^2\alpha^2}\alpha^{d-2}d\alpha\leq C\int_0^\infty e^{-cu^2}u^{d-2}du.$$

Having proved (11) and (12), by changing the order of integration in (9) and (10) we get also the desired estimate for the terms involving I_0 and I_1, respectively, and at the same time we end the sketch of the proof.

Even though there were different proofs of the weak type $(1,1)$ for the Ornstein–Uhlenbeck maximal operator, the question of majorizing it by an appropriate maximal function remained open. Very recently, Aimar, Forzani, and Scotto [1] (see also [6]) were able to bound the diffusion maximal operator by the maximal function defined by

$$M_\Phi f\ (x)=\sup_{0<r<1}\frac{1}{\gamma((1+\delta)B(\frac{x}{r},\frac{|x|}{r}(1-r)))}\int_{\mathcal{R}^d}\Phi\Big(\frac{|x-r\,y|}{\sqrt{1-r^2}}\Big)|f(y)|\ d\gamma(y), \tag{14}$$

where $\Phi:\mathcal{R}_o^+\to\mathcal{R}_o^+$; Φ is non-increasing, $S=\sum_{\nu\geq 1}\Phi(\frac{1}{2}(\nu-1))\,\nu^{2d}<\infty$, and $\delta=\delta_{r,x}=\frac{r}{|x|(1-r)}\min\{\frac{1}{|x|},\sqrt{1-r}\}$.

By taking $\Phi\ (t)=\frac{1}{\pi^{d/2}}\exp(-t^2)$, they get that

$$W^*f(x)\leq CM_\Phi f(x)$$

and prove the weak type $(1,1)$ of this new operator by using a covering lemma, halfway between the Besicovitch and Wiener covering lemmas.

2.2 Riesz transforms

Now let us turn to the study of the Riesz transforms. According to definition (1) in the Introduction, there are two ways of generating higher order Riesz transforms. One way is by always using the same derivative (using $(D_{\mathcal{L}})_i = \frac{\partial}{\partial x_i}$ or $(D^*_{\mathcal{L}})_i = -\frac{\partial}{\partial x_i} + 2x_i$) and the other is by mixing both derivatives. So far only the first class of higher order Riesz transforms has been studied; to our knowledge, nothing is known about the second class.

First we deal with Riesz transforms involving the derivative $(D_{\mathcal{L}})_i$.

Many authors have proved the strong boundedness of these Riesz transforms on $L^p(d\gamma)$ for $1 < p < \infty$.

As a pioneer work we can mention Muckenhoupt (1969) [25] for the first order operator in $d = 1$. Meyer (1984) [23] and later Gundy (1986) [16] gave probabilistic proofs valid for any order and any dimension, with a constant independent of dimension and thus also true for the infinite-dimensional case. From an analytic point of view, we have Pisier's proof (1988) [30], using Calderón's method of rotation, valid for operators of odd order and with constant independent of dimension. Another analytic proof was given by Urbina (1990) in [41] where, following Muckenhoupt's idea, he splits the kernel into its local and global parts; for the first one he shows that it is a Calderón–Zygmund kernel, while in the global region he bounds the operator by a composite of one-dimensional maximal functions. In this way he gives a proof valid for any order and any dimension. By using the Littlewood–Paley–Stein g functions, Gutiérrez (1994) in [17] also proved the $L^p(d\gamma)$ boundedness of the first order operators for $1 < p < \infty$, with constant independent of dimension, which was later extended to Riesz transforms of any order by Gutiérrez, Segovia, and Torrea (1996) in [18]. We will give more details of that proof in Section 2.3. Finally, this result for higher order Riesz transforms with constant independent of dimension was also proved by Forzani, Scotto, and Urbina (2001) [11] based on the boundedness of the Riesz transforms of order one. They showed that these higher order operators are related to the composite of first order Riesz transforms through a multiplier of Meyer type.

Surprisingly, the higher order Riesz transforms defined through the derivative $(D_{\mathcal{L}})_i$ need not be weak type $(1,1)$ for all orders. Indeed, they are if and only if their order is at most two. The weak type $(1,1)$ for the Riesz of order one and $d = 1$ is due to Muckenhoupt (1969) [25] and to Fabes, Gutiérrez, and Scotto (1994) [5] for $d > 1$. The weak type $(1,1)$ for order two in dimension one was proved by Forzani and Scotto (1998) [9], and a proof in higher dimensions appeared in Pérez's doctoral dissertation [26] (see also [29]). For order greater than two, a counter-example in one dimension is given in [9] and in [13] for higher dimensions due to García-Cuerva, Mauceri, Sjögren, and Torrea (1999).

To prove the positive results regarding the weak type $(1,1)$, the operators are again split into a local part, and a global one. On the local part, these operators are written as a singular integral with a Calderón–Zygmund kernel plus an operator of strong type $(1,1)$. It also turns out that for Calderón–Zygmund operators, if they are of weak type $(1,1)$ with respect to the Lebesgue measure, they can be proved to be weak type $(1,1)$ with respect to the Gaussian measure, once restricted to the local zone. This fact was proven by Muckenhoupt in [25] in dimension one and extended to higher dimensions by Fabes, Gutiérrez, and Scotto in [5]. As for the global region, concerning the Riesz transforms of order one, Fabes, Gutiérrez, and Scotto divided it into five subregions, following Sjögren's forbidden region technique. They obtain slightly bigger bounds, but they are still good enough to get the weak type $(1,1)$.

Instead, Pérez and Soria bounded the global region of the kernels for Riesz transforms up to order two by the positive operator whose kernel is $K^*(x,y)$ defined in (6). Then, they showed that this last operator is weak type $(1,1)$ with respect to the Gaussian measure with a technique different from the one given in Section 2.1.

On the other hand, Riesz transforms defined through the derivative $(D^*_{\mathcal{L}})_i$ are bounded on $L^p(d\gamma)$, $1 < p < \infty$, and weak type $(1,1)$ independently of their order, as was proved by Aimar, Forzani, and Scotto (2007) [1]. The boundedness of the local part follows the pattern we have just described. For the global part, they bounded the kernels of the Riesz transforms of any order by the maximal function defined in (14) with $\Phi(t) = e^{-ct^2}$. As we mentioned, this maximal function turns out to be weak type $(1,1)$ with respect to the Gaussian measure. Let us point out that it is also possible to bound the kernels of these new Riesz transforms by K^* as defined in (6), and the weak type $(1,1)$ would follow from there as well.

2.3 The Littlewood–Paley–Stein functions

These functions are a very useful tool in classical harmonic analysis. A well-known application consists of getting the boundedness on $L^p(dx)$, $1 < p < \infty$, for the Riesz transforms with constant independent of dimension. Within the context of Gaussian harmonic analysis, Gutiérrez in [17] and Gutiérrez, Segovia, and Torrea in [18], using appropriate Littlewood–Paley–Stein g functions, obtained free-dimensional $L^p(d\gamma)$ estimates for the first and higher order Riesz transforms, respectively.

The method is interesting for various reasons. As far as we know, it is the only analytic proof where the differential equation involved is used, and also it lets them get a constant uniform on dimension.

Now we sketch the proof for the first order Riesz transforms following the technique used in [17] and [18]. Let $P^1_t f = e^{-t\sqrt{\mathcal{L}+2I}} f$ be a modification of the Poisson semigroup $P_t f = e^{-t\sqrt{\mathcal{L}}} f$. For the case here, it is necessary to consider two g functions

$$g(f)(x) = \left(\int_0^\infty |t \nabla P_t f(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{15}$$

where $\nabla = \left(\frac{\partial}{\partial t}, \nabla_x\right)$ and

$$g_1(f)(x) = \left(\int_0^\infty |t\frac{\partial}{\partial t}P_1^t f(x)|^2 \frac{dt}{t}\right)^{1/2}. \tag{16}$$

The relation between these operators comes from the fact that

$$\frac{\partial P_t f(x)}{\partial x_i} = \frac{\partial P_t^1(R_i f)(x)}{\partial t}. \tag{17}$$

Using (17) on (15) and (16), the authors get the inequality

$$\sum_{i=1}^{d}(g_1(R_i f)(x))^2 \leq (g(f)(x))^2,$$

and therefore

$$(g_1(R_i f)(x))^2 \leq (g(f)(x))^2.$$

From these inequalities, the result follows if one is able to prove that for $1 < p < \infty$, there is a constant C_p, independent of d, such that for every $f \in L^p(d\gamma)$

$$\|g(f)\|_p \leq C_p\|f\|_p \tag{18}$$

and

$$\|f\|_p \leq C_p\|g_1(f)\|_p. \tag{19}$$

Now in order to prove (18), by using the differential equation that P_t satisfies, it is possible to write ∇P_t as

$$|\nabla P_t f(x)|^2 = C|P_t f|^{(2-p)}\bar{\mathcal{L}}(P_t f)^p, \tag{20}$$

where $\bar{\mathcal{L}} = \frac{\partial^2}{\partial t^2} - \mathcal{L}$. The first factor, when $1 < p \leq 2$, is majorized by means of the maximal operator P^*, which is known to be bounded on $L^q(d\gamma)$ for $1 < q \leq \infty$. As for the second factor, by using the differential equation again, it is proved that

$$\int_{\mathcal{R}^d}\int_0^\infty t\ \bar{\mathcal{L}}(P_t f)^p\ dtd\gamma \leq C||f||_p. \tag{21}$$

Then, as a consequence of (20), (21), and Hölder's inequality, the $L^p(d\gamma)$ boundedness of g for $1 < p \leq 2$ follows.

Next, the result is extended for $4 \leq p < \infty$, using the key estimate

$$\int_{\mathcal{R}^d} g(f)^2(x)\ \phi(x)\ d\gamma(x) \leq C_p\left(\int_{\mathcal{R}^d} f(x)^2\ \phi(x)\ d\gamma(x)\right.$$
$$\left.\int_{\mathcal{R}^d} P^*f(x)\ g(f)(x)\ g(\phi)(x)\ d\gamma(x)\right),$$

and what is known for $1 < p \leq 2$.

Using interpolation the inequality (18) also holds for the remaining values of p.

With the same method, the boundedness of g_1 on $L^p(d\gamma)$ is also obtained. This latter result is then used to prove (19). It is not hard to check the following identity:

$$4\|g_1(f)\|_2^2 = \|f\|_2^2,$$

and by polarization we have for $f, h \in L^2(d\gamma)$

$$4\int_0^\infty \int_{\mathcal{R}^d} t\frac{\partial}{\partial t}P_t^1 f \;\frac{\partial}{\partial t}P_t^1 h \; d\gamma dt = \int_{\mathcal{R}^d} f\; h\; d\gamma. \tag{22}$$

Then, inequality (19) follows essentially by taking f in $L^p(d\gamma) \cap L^2(d\gamma)$, a dense subspace of $L^p(d\gamma)$, calculating out $\|f\|_p$ through duality, using the identity (22), Schwarz's inequality on t, Hölder's inequality on x, and the boundedness of g_1 on $L^{p'}(d\gamma)$.

All these results are then extended to the Riesz transforms of any order by using the operators $P_t^k = e^{-t\sqrt{\mathcal{L}+2kI}}$ and their corresponding g_k functions. In the proof, Gutiérrez, Segovia, and Torrea use a clever extension of the Littlewood–Paley–Stein theory of g functions to the vector-valued setting and the strong type inequalities for the Riesz transforms of order one.

Later on, Pérez in [27] and Forzani, Scotto, and Urbina in [10] proved that g functions involving the space derivatives behave exactly the same as the corresponding Riesz transforms, when acting on $L^1(d\gamma)$. They are of weak type $(1,1)$ if and only if the order is at most two. Meanwhile, the higher order g functions with respect to time derivatives are all weak type $(1,1)$. To our knowledge, nothing is known when $D_{\mathcal{L}}^*$ is used instead of $D_{\mathcal{L}}$.

3 Hermite functions

The study of the operators related to Hermite functions turns out to be much simpler than that of those related to Hermite polynomial expansions. On the one hand the operator L_d has all its eigenvalues strictly positive and, on the other hand, the measure is just the Lebesgue measure.

In the sequel, for a multi-index α, we shall denote by h_α the normalized Hermite function defined by $h_\alpha(x) = \mathcal{H}_\alpha(x)e^{-|x|^2/2}$, where $\mathcal{H}_\alpha$ is the normalized Hermite polynomial on $L^2(d\gamma)$. These functions satisfy

$$L_d h_\alpha = (2|\alpha| + d)\; h_\alpha.$$

As we mentioned in the Introduction, finite linear combinations of Hermite functions are dense on $L^p(dx)$, $1 \le p < \infty$, even if we consider weighted Lebesgue spaces with weights in the A_p-class of Muckenhoupt, as is shown in [37]. Such a result is a consequence of some precise bounds for the Hermite functions given by Askey and Wainger, [2], that allow us to show that L^p-norms of Hermite functions grow polynomially with the order of the functions.

Also, such estimates are the key in proving that for each operator given in the Introduction, the spectral definition coincides with its integral version. Many of the details are worked out in the neat exposition of Stempak and Torrea [37].

It is worth mentioning here that Hermite functions partial sums of a function in $L^p(dx)$ rarely converge. In fact, when $d = 1$ it only happens for $\frac{4}{3} < p < 4$, [2], and in higher dimensions only for $p = 2$, [40].

Along this section the notation $\| \|_p$ stands for the L^p-norm taken with respect to the Lebesgue measure.

3.1 The maximal operator for the heat-diffusion semigroup

It can be easily seen that $T^*f = \sup_{t>0} |e^{-tL_d} f|$ is bounded pointwise by the Hardy–Littlewood maximal function.

In fact, if in the expression of the kernel given in (3) we perform the change of parameter $t = t(s) = \frac{1}{2} \log \frac{1+s}{1-s}$, $0 \le s \le 1$, already used in [37], we are led to

$$W_{L_d}(t,x,y) = \left(\frac{1-s^2}{4\pi s}\right)^{d/s} e^{-\frac{1}{4s}|x-y|^2 - \frac{s}{4}|x+y|^2},\ t = t(s).$$

From here we clearly have

$$W_{L_d}(t,x,y) \le (1-s^2)^{d/2} \mathcal{W}_s(x-y),\ t = t(s),$$

where $\mathcal{W}_s(x-y)$ is the classical heat kernel. This inequality shows that T_t is a contraction semigroup, and moreover for the maximal operator we have

$$T^*f(x) \le Mf(x),$$

where M denotes the Hardy–Littlewood maximal function. As a consequence, L^p boundedness for $p > 1$ and weak type $(1,1)$ are obtained for any weight belonging to the corresponding Muckenhoupt class.

3.2 Riesz transforms

As we already mentioned, Riesz transforms for Hermite functions were first introduced by Thangavelu [40]. Following his notation we write

$$R_j^{\pm} = A_j^{\pm} L^{-1/2},\ \ j = 1, \dots, d,$$

where $A_j^{\pm} = \frac{\partial}{\partial x_j} \pm x_j$.

The action of these Riesz transforms over h_α is given by

$$R_j^+ h_\alpha = \left(\frac{2\alpha_j}{2|\alpha| + d}\right)^{1/2} h_{\alpha - e_j},\ \ R_j^- h_\alpha = \left(\frac{2\alpha_j + 2}{2|\alpha| + d}\right)^{1/2} h_{\alpha + e_j},$$

where e_j denotes the multi-index with all zeros except for a one in the jth coordinate. Clearly, this action can be extended to $L^2(dx)$, producing bounded operators.

One of the classical approaches in dealing with the Riesz transforms is through the notion of conjugate Poisson integrals. In the Hermite functions context, that notion was introduced by Thangavelu setting

$$u_j^{\pm}(x,t) = P_t R_j^{\pm} f(x), \;\; j = 1, \dots, d$$

as the conjugates of $u(x,t) = P_t f(x)$.

To extend Riesz transforms to $L^p(dx)$, Thangavelu shows that for such functions f, $\lim_{t\to 0^+} u_j^{\pm}(x,t)$ exists and it generates bounded operators on $L^p(dx)$, $1 < p < \infty$, and of weak type for $p = 1$. This kind of approach was also used in [15] for $d = 1$.

On the other hand, Stempak and Torrea [37] show that the corresponding kernels as given in the Introduction, i.e.,

$$\mathcal{R}_j^{\pm}(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{\partial}{\partial x_j} \pm x_j \right) W_{L_d}(t,x,y) t^{-1/2} dt,$$

are indeed singular integral kernels associated to the above Riesz transforms in the sense of Calderón–Zygmund, namely

$$\int_{\mathcal{R}^d} R_j^{\pm} f(x) \; g(x) dx = \int_{\mathcal{R}^d} \int_{\mathcal{R}^d} \mathcal{R}_j^{\pm}(x,y) f(y) dy \; g(x) dx$$

for any $f, g \in C_o^\infty(\mathcal{R}^d)$ with disjoint supports.

Moreover, they checked that such kernels and their partial derivatives have the right size to be Calderón–Zygmund kernels.

In this way, as a consequence of the general Calderón–Zygmund theory, they obtain the boundedness of Riesz transforms on $L^p(w)$, $1 < p < \infty$, $w \in A_p$-class of Muckenhoupt and weak type $(1,1)$ for the weight in A_1.

In the same paper, the authors also show that for functions in those weighted spaces their conjugates are well defined (in fact they are C^∞-functions), and moreover

$$\lim_{t\to 0^+} u_j^{\pm}(x,t) = R_j^{\pm} f(x) \quad \text{a.e.}$$

for $1 < p < \infty$.

Recently, in [39] Stempak and Torrea have extended their results to higher order Riesz transforms.

Let us remark that either with Thangavelu's or Stempak and Torrea's approach, the operator norm of Riesz transforms acting on $L^p(dx)$, $1 < p < \infty$, might depend on the dimension d.

In [19] Carlos Segovia et al., with the aid of an appropriate Littlewood–Paley–Stein g function theory, are able to prove boundedness on $L^p(dx)$, $1 < p < \infty$, with constants independent of dimension. Their method allows them to extend such a result to higher order Riesz transforms as well.

3.3 Littlewood–Paley–Stein g functions

Thangavelu in [40], Chapter 4, considered some g functions with respect to the heat-diffusion semigroup in the context of Hermite functions. They are defined as

$$g_k(f)(x) = \left(\int_0^\infty |\frac{\partial^k}{\partial t^k} T_t f(x)|^2 t^{2k-1} dt \right)^{\frac{1}{2}}.$$

He proves boundedness on $L^p(dx), 1 < p < \infty$, and weak type $(1,1)$ even with A_p-weights. He used them in order to get a kind of Marcinkiewicz's multiplier theorem for Hermite function expansions.

As we pointed out in the previous section, in [19], the authors also considered g functions, although this time with respect to the Poisson semigroup and involving not only derivatives with respect to t but also all the other *partial derivatives* $A_j^{\pm}, j = 1, \ldots, d$. Using a different approach, similar to the one described for the polynomial case, they obtained L^p-boundedness for $1 < p < \infty$. Their technique makes strong use of the fact that $u(x,t) = P_t f(x)$ satisfies a Laplace-type equation, namely

$$\frac{\partial^2 u}{\partial t^2} - L_d u = 0.$$

In fact, they make this analysis for g functions associated to the more general differential operator $L_d^b = L_d + b$ with b any integer. Such a degree of generality is needed in order to get dimension-free estimates for the Riesz transforms on $L^p(dx), 1 < p < \infty$. By using Krivine's result [21], they are able to extend their boundedness results to higher order g functions, mixing any kind of derivatives.

In a recent paper, Stempak and Torrea have obtained weighted inequalities for the above g functions of any order. Boundedness on $L^p(w), 1 < p < \infty$, is proved for w in the A_p-class together with weak type $(1,1)$ for w in A_1. The authors achieve these results by taking advantage of the vector-valued Calderón–Zygmund theory (see [38]).

Acknowledgements

We are grateful to Carlos Segovia for numerous enjoyable and enlightening discussions on these topics and on many others. We feel though that his stimulating influence in our lives goes far beyond mathematics.

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Weights for One–Sided Operators

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Summary. We present a survey about weights for one-sided operators, one of the areas in which Carlos Segovia made significant contributions. The classical Dunford–Schwartz ergodic theorem can be considered as the first result about weights for the one-sided Hardy–Littlewood maximal operator. From this starting point, we study weighted inequalities for one-sided operators: positive operators like the Hardy averaging operator, the one-sided Hardy–Littlewood maximal operator, singular approximations of the identity, one-sided singular integrals. We end with applications to ergodic theory and with some recent results in dimensions greater than 1.

Key words: Weight, one-sided operator, ergodic theorem.

1 Introduction

One of the many areas of harmonic analysis in which Carlos Segovia has made significant contributions is the theory of weights for one-sided operators. As far as we know, one cannot find in the literature a more or less complete account of the main results and applications of this theory. These pages are an attempt to fill this gap and have been written to honor the memory of Carlos Segovia, who died while we were in the process of writing them.

First of all, let us define our terms: By a weight w we mean a locally integrable nonnegative function, and we say that an operator T acting on measurable functions on an interval (a, b), $-\infty \leq a < b \leq \infty$, in $\mathbb{R}$ or $\mathbb{Z}$, is one sided if the value of $Tf(x)$ only depends on the values of f in $(a, x]$ or $[x, b)$; that is, $Tf(x) = T(f\chi_{[x,b)})(x)$ for all x or $Tf(x) = T(f\chi_{(a,x]})(x)$ for all x. One easy example is $Tf(x) = \int_0^x f(t)\,dt$ for functions in $(0, \infty)$. The type of problems we will be interested in are the following: Given a one-sided operator T, characterize if possible the pairs of weights (u, v) such that we have

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*,
Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_6,

$$\left(\int |Tf|^q u\right)^{1/q} \leq C\left(\int |f|^p v\right)^{1/p} \tag{1}$$

or at least the weaker inequality

$$\left(\int_{\{x:|Tf(x)|>\lambda\}} u\right)^{1/q} \leq \frac{C}{\lambda}\left(\int |f|^p v\right)^{1/p}, \quad \lambda > 0. \tag{2}$$

Inequalities (1) and (2) are called, respectively, weighted weak type and weighted strong type (p,q) inequalities for T with weights u and v. One-sided operators appear in a natural way in harmonic analysis and ergodic theory. Weighted inequalities for one-sided operators are connected to the following classical theorem.

Theorem 1.1 (*Dunford–Schwartz*) *Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{T} = \{T^t : t \in \mathbb{R}, t > 0\}$ be a strongly measurable semi-group of operators in $L^1(X, \mathcal{F}, \mu)$ with $\|T^t\|_1 \leq 1$ and $\|T^t\|_\infty \leq 1$. Let*

$$M_{\mathcal{T}} f(x) = \sup_{h>0} \frac{1}{h}\left|\int_0^h T^t f(x)\, dt\right|.$$

Then there is an absolute constant C such that for any $\lambda > 0$ and all $f \in L^1(\mu)$

$$\mu(\{x \in X : M_{\mathcal{T}} f(x) > \lambda\}) \leq \frac{C}{\lambda}\int_X |f|\, d\mu.$$

We claim that this theorem gives information about weighted inequalities for one-sided operators. Indeed, if we take $X = \mathbb{R}$, with measure $w(x)\,dx$, and $T^t f(x) = f(x+t)$, then the ergodic maximal function is the one-sided Hardy–Littlewood maximal operator

$$M^+ f(x) = \sup_{h>0} \frac{1}{h}\left|\int_x^{x+h} f(t)\, dt\right| .$$

T^t are obviously contractions in $L^\infty(w)$, and it is very easy to see that they are contractions in $L^1(w)$ if and only if w is increasing. In other words, any increasing function is a good weight for the one-sided Hardy–Littlewood maximal operator.

This is in sharp contrast to the two-sided case. Let us recall that if one considers the two-sided Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{h,k} \frac{1}{h+k}\left|\int_{x-k}^{x+h} f\right|,$$

then (2) holds for $T = M$, $u = v = w$, $p = q = 1$ if, and only if, w satisfies the Muckenhoupt A_1 condition, that is, there exists C such that

$$Mw(x) \leq Cw(x) \quad \text{a.e.}$$

It is easy to see that this condition implies that the weight is doubling; i.e., there exists C such that for any interval I if we denote by $2I$ the interval concentric with I but twice as long, then $\int_{2I} w \leq C \int_I w$, which of course is false for the increasing function $w(x) = e^x$. Therefore, the good weights for M^+ are a wider class than the good weights for M.

In the remaining seven sections we will present a survey of the main results about weights for one-sided operators. The first five sections deal with positive operators. In §2 we present results for the Hardy averaging operator $Pf(x) = \frac{1}{x}\int_0^x f(y)\,dy$; §3 is devoted to the characterization of the weights for which (1) and (2) hold when T is M^+; in §4 we study properties of these weights and show that there are substitutes for the doubling condition and for A_∞, which are good enough to prove the $p - \epsilon$ property and good lambda inequalities. As in the two-sided case, these ideas allow us to study (one-sided) fractional integrals relating them to the one-sided fractional maximal operator $M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f|$. In §5 we consider nonstandard approximations of the identity, while §6 presents results for one-sided singular integrals, divided in three parts: strongly singular integrals, Calderón–Zygmund operators, and other singular integrals. §7 presents some applications to ergodic theory. We end with a section about the situation in dimensions greater than 1.

Throughout the chapter we shall use standard notation. In particular, if $1 < p < \infty$ then p' stands for its conjugate exponent.

2 Weighted Hardy inequalities

One of the simplest one-sided operators is the Hardy averaging operator P defined for functions f on $(0, \infty)$ by

$$Pf(x) = \frac{1}{x}\int_0^x f.$$

G. H. Hardy [24], in 1920, was the first author who proved weighted inequalities for P. Specifically, Hardy showed that if $p > 1$ and $\varepsilon < p - 1$, then the inequality

$$\int_0^\infty Pf(x)^p x^\varepsilon dx \leq K \int_0^\infty f(x)^p x^\varepsilon dx$$

holds for all positive functions f on $(0, \infty)$ with a constant K independent of f. The above inequality is a strong type (p, p) inequality with power weights for the operator P. This result was the origin of the theory of Hardy inequalities. Starting at this point, it is natural to consider the problem of determining the pairs of nonnegative measurable functions (u, v) for the operator P to verify the weak or strong type (p, q) inequality with weights u and v. With regard to the weighted strong type inequality, the characterization in the case $1 \le p \le q < \infty$ is due to J. Bradley [12], who improved previous results due to G. Talenti [59], G. Tomaselli [60] and B. Muckenhoupt [42], whereas V. Maz'ya [41] solved the problem in the case $1 \le q < p < \infty$. The next theorem collects Bradley's and Maz'ya's results:

Theorem 2.1 *([12, 41]) The operator P verifies the strong type (p,q) inequality with weights u and v if and only if $J < \infty$, where*

$$J = \sup_{t>0} \left(\int_t^\infty \frac{u(x)}{x^q} dx \right)^{\frac{1}{q}} \left(\int_0^t v(x)^{1-p'} dx \right)^{\frac{1}{p'}} \qquad \text{if } p \le q$$

and

$$J = \int_0^\infty \left(\int_t^\infty \frac{u(x)}{x^q} dx \right)^{\frac{r}{q}} \left(\int_0^t v(x)^{1-p'} dx \right)^{\frac{r}{q'}} v(t)^{1-p'} dt \qquad \text{if } q < p,$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

The study of the weighted weak type inequalities for P was undertaken by K. Andersen and B. Muckenhoupt [3] in 1982. They worked with the operators P_α defined by

$$P_\alpha f(x) = x^\alpha \int_0^x f,$$

which are called modified Hardy operators. Their result reads as follows.

Theorem 2.2 *([3]) Let $1 \le p \le q < \infty$ and let u and v be nonnegative measurable functions.*

(i) If $\alpha \ge 0$, the weighted weak type inequality with weights u and v for P_α holds if and only if

$$B(\alpha) = \sup_{t>0} t^\alpha \left(\int_t^\infty u \right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

(ii) If $\alpha < 0$, the weighted weak type inequality with weights u and v for P_α holds if and only if there exists $s > 0$ such that

$$B(\alpha, s) = \sup_{t>0} \left(\int_t^\infty \frac{t^s}{x^{s-\alpha q}} u(x) dx \right)^{\frac{1}{q}} \left(\int_0^t v(x)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

The weighted weak type inequalities for more general modified Hardy operators were studied by E. Ferreyra [20] and F. J. Martín-Reyes, P. Ortega, and M. D. Sarrión [35]. They considered the operator

$$P_g f(x) = g(x) \int_0^x f,$$

where g is a fixed nonnegative measurable function, and proved the following result.

Theorem 2.3 (*[20], [35]*) *Let $1 \le p \le q < \infty$. The operator P_g verifies the weighted weak type (p,q) inequality with weights u and v if and only if*

$$\sup_{t>0} \|g\chi_{(t,\infty)}\|_{q,\infty;w} \left(\int_0^t v^{1-p'}\right)^{\frac{1}{p'}} < \infty.$$

Ferreyra applied some duality techniques based on a paper by E. Sawyer [48]. However, these techniques do not work for characterizing the weighted weak type $(1,1)$ inequality. This problem was solved in [35] as a particular case of the results for modified Hardy operators in generalized Lorentz spaces $\Lambda_u^{p,q}(w)$.

The characterization of the weighted weak type inequality for P_g in the case $1 \le q < p$ is due to Martín-Reyes and Ortega [34]. They worked with monotone g and proved the following result.

Theorem 2.4 (*[34]*) *Let $1 \le q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and assume that g is a monotone function. The weighted weak type (p,q) inequality with weights u and v for P_g holds if and only if the function Φ, defined on $(0,\infty)$ by*

$$\Phi(x) = \sup_{d>x} \left[\inf_{y\in(x,d)} g(y) \left(\int_x^d u\right)^{\frac{1}{p}} \left(\int_0^x v^{1-p'}\right)^{\frac{1}{p'}} \right],$$

belongs to the space $L^{r,\infty}(u)$.

It is an open problem to find a reasonable characterization of the weighted weak type (p,q) inequality for P_g with general g in the case $q < p$.

It is worth noting that all the results cited above can be stated and proved, with obvious changes in the characterizing conditions, if the interval $(0,\infty)$ is replaced by an interval (a,b) with $-\infty < a < b < \infty$. In particular, the Hardy averaging operators in an interval (a,b) are the operators

$$P^+ f(x) = \frac{1}{b-x}\int_x^b f \qquad \text{and} \qquad P^- f(x) = \frac{1}{x-a}\int_a^x f.$$

As we have just seen, the problem of the characterization of the weak and strong type inequalities with weights for the one-dimensional modified Hardy operators is almost completely solved. However, the situation is very different for higher dimensions. The n-dimensional modified Hardy operator is defined for functions f on $(0,\infty)^n = (0,\infty)\times\cdots\times(0,\infty)$ by

$$T_g f(x) = g(x)\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} f,$$

where $x = (x_1,\ldots,x_n)$. This operator is one sided in $\mathbb{R}^n$ in the sense that the value of $T_g f(x)$ only depends on the values of f over the points y such that $0 \le y_i \le x_i$ for all $i = 1,2,\ldots,n$. E. Sawyer [49] is perhaps the only author who has obtained some results for T_g. He has worked in dimension 2 with $g = 1$ and has characterized the weighted weak and strong type (p,q) inequalities for T_g in the case $1 < p \le q < \infty$. His result is the following.

Theorem 2.5 *([49]) Let $1 < p \le q < \infty$ and let u and v be nonnegative measurable functions on $(0,\infty)^2 = (0,\infty)\times(0,\infty)$. Then,*

(i) there exists $C > 0$ such that the inequality

$$\left(\int_0^\infty \int_0^\infty \left(\int_0^{x_1}\int_0^{x_2} f\right)^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \le C\left(\int_0^\infty\int_0^\infty f^p v\right)^{\frac{1}{p}}$$

holds for all nonnegative measurable functions f on $(0,\infty)^2$ if and only if the following three conditions hold:

$$\sup_{y_1,y_2>0}\left(\int_{y_1}^\infty\int_{y_2}^\infty u\right)^{\frac{1}{q}}\left(\int_0^{y_1}\int_0^{y_2} v^{1-p'}\right)^{\frac{1}{p'}} < \infty, \tag{3}$$

$$\sup_{y_1,y_2>0}\frac{\left(\int_0^{y_1}\int_0^{y_2}\left(\int_0^{x_1}\int_0^{x_2} v^{1-p'}\right)^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int_0^{y_1}\int_0^{y_2} v^{1-p'}\right)^{\frac{1}{p}}} < \infty \tag{4}$$

and

$$\sup_{y_1,y_2>0}\frac{\left(\int_{y_1}^\infty\int_{y_2}^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty u\right)^{p'} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p'}}}{\left(\int_{y_1}^\infty\int_{y_2}^\infty u\right)^{\frac{1}{q'}}} < \infty; \tag{5}$$

(ii) there exists $C > 0$ such that the inequality

$$\left(\int_{\{(x_1,x_2)\in(0,\infty)^2:\int_0^{x_1}\int_0^{x_2} f>\lambda\}} u\right)^{\frac{1}{q}} \le \frac{C}{\lambda}\left(\int_0^\infty\int_0^\infty f^p v\right)^{\frac{1}{p}}$$

holds for all $\lambda > 0$ and all nonnegative measurable functions f on $(0,\infty)^2$ if and only if (3) and (5) hold.

There are no general results either in higher dimensions or in the case $q < p$.

The theory of weighted Hardy inequalities has been developed following many directions. We have only presented the classical results of the theory and some others that will be used in the next section. The interested reader can find more complete expositions in the books [46], [31], and [30].

3 Weights for the one-sided Hardy–Littlewood maximal operators

Two classical operators closely related to the Hardy averaging operators are the one-sided Hardy–Littlewood maximal operators M^+ and M^- defined for functions of one real variable by

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \qquad \text{and} \qquad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|.$$

The relationship we have just referred to allows us to reduce the characterization of the weighted weak type inequalities for M^+ and M^- to the ones for the Hardy averaging operators. This approach appears in a paper by E. Sawyer [51], although it is due to K. Andersen (see [51]). It is based on the Riesz rising sun lemma which ensures that the connected components (a_j, b_j) of the open set $O_\lambda = \{x \in \mathbb{R} : M^-f(x) > \lambda\}$ verify

$$\frac{1}{x - a_j} \int_{a_j}^{x} |f| > \lambda$$

for all $x \in (a_j, b_j)$. Then, if u and v are nonnegative measurable functions, we have

$$\begin{aligned}\int_{\{x\in\mathbb{R}:M^-f(x)>\lambda\}} u &= \sum_j \int_{a_j}^{b_j} u = \sum_j \int_{\{x\in(a_j,b_j):\frac{1}{x-a_j}\int_{a_j}^{x}|f|>\lambda\}} u \\ &\le \frac{C}{\lambda^p} \sum_j \int_{a_j}^{b_j} |f|^p v \le \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p v,\end{aligned}$$

where the first inequality holds with a constant independent of f and λ if all the operators $P_j^- f(x) = \frac{1}{x-a_j}\int_{a_j}^{x} f$ are of weak type (p, p) uniformly with respect to the weights u and v. In this way, Sawyer established the following result.

Theorem 3.1 (*[51]*) *Let $p \ge 1$. The following statements are equivalent.*

(i) There exists $C > 0$ such that the inequality

$$\int_{\{x\in\mathbb{R}:M^-f(x)>\lambda\}} u \le \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p v$$

holds for all f and all $\lambda > 0$.

(ii) The pair (u, v) verifies the condition A_p^-, which means

$$M^+u(x) \le Cv(x) \ \text{a.e. if } p = 1$$

and

$$\sup_{a<b<c} \frac{1}{c-a} \left(\int_b^c u\right)^{\frac{1}{p}} \left(\int_a^b v^{1-p'}\right)^{\frac{1}{p'}} < \infty \ \text{if } p > 1.$$

Observe that, in accordance with Theorems 2.2 and 2.3, the condition A_p^- expresses the fact that the operators $P_{a,c}f(x) = \frac{1}{x-a}\int_a^x f$, $x \in (a, c)$, are all of weak type (p, p) uniformly with respect to the weights u and v.

Sawyer also characterized the two-weight strong type inequality for M^-.

Theorem 3.2 *([51]) Let $p > 1$ and let u and v be weights. The following statements are equivalent.*

(i) There exists $C > 0$ such that

$$\int_{\mathbb{R}} (M^- f)^p u \le C \int_{\mathbb{R}} |f|^p v$$

for all $f \in L^p(v)$.

(ii) The pair (u, v) verifies the condition S_p^-, i. e., there exists $C > 0$ such that

$$\int_I (M^-(v^{1-p'}\chi_I))^p u \le C \int_I v^{1-p'} < \infty$$

for all bounded intervals $I = (a, b)$ with $\int_b^\infty u < \infty$.

Subsequently, Sawyer observed that for $u = v$ and $p > 1$ the conditions A_p^- and S_p^- are equivalent. The argument follows the line of [25], but conveniently adapted to the one-sided setting. This observation together with Theorems 3.1 and 3.2 gives immediately that if $u = v$ and $p > 1$ then the weighted weak type inequality for M^-, the weighted strong type inequality for M^-, and the condition A_p^- are all equivalent.

Similar results can be obtained for the operator M^+ changing conditions A_p^- by A_p^+, which means

$$M^- u(x) \le C v(x) \text{ a.e. if } p = 1$$

and

$$\sup_{a<b<c} \frac{1}{c-a} \left(\int_a^b u \right)^{\frac{1}{p}} \left(\int_b^c v^{1-p'} \right)^{\frac{1}{p'}} < \infty \text{ if } p > 1.$$

Together with the characterizations of the weighted inequalities for M^+ and M^-, Sawyer obtained in [51] some properties of the weights belonging to the classes A_p^+ and A_p^-. Specifically, he proved:

(i) If $w \in A_1^+$, then there exists $\delta > 0$ such that $w^{1+\delta} \in A_1^+$.
(ii) (Factorization of weights) If $p > 1$, $w \in A_p^+$ if and only if there exist $w_0 \in A_1^+$ and $w_1 \in A_1^-$ such that $w = w_0 w_1^{1-p}$.
(iii) If $p > 1$ and $w \in A_p^+$, then there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}^+$.

These results are the one-sided versions of the well-known results about Muckenhoupt's classes A_p. However, Sawyer's proof of property (iii) was not direct. He obtained (iii) by combining (i) and (ii), which in turn depends on the characterization of the one-weight strong type inequality for M^+.

New proofs of the characterizations of the weighted weak type inequality and the one-weight strong type inequality for the one-sided maximal operators were given in [32]. On one hand, in [32] a Whitney-type decomposition of the connected components of the set $O_\lambda = \{x : M^+ f(x) > \lambda\}$ was used in order to characterize the weighted weak type inequality for M^+. On the other hand, some variants of the above-mentioned technique were applied to show that if $p > 1$ and $w \in A_p^+$ then w verifies a weak reverse Hölder inequality. With this tool, a direct proof of the property $A_p^+ \Rightarrow A_{p-\varepsilon}^+$ was given in [32]. This allows us to show immediately, by interpolation, that for $p > 1$ the one-weight strong type inequality and the one-weight weak type inequality for the one-sided maximal operators are equivalent. In the next section we shall check some properties of the A_p^+ weights, and we shall go back to the weak reverse Hölder inequality.

4 Some remarks and properties of the one-sided weights

4.1 Basic weights

As we said above, a weight $w \in A_p^+$ if and only if there exist weights $u \in A_1^+$ and $v \in A_1^-$ such that $w = uv^{1-p}$. The basic weights in A_1^+ are the increasing functions. In fact, given a weight $w \in A_1^+$ and an interval I, there exists an increasing function g_I on I such that $\int_I w \le C \int_I g_I$ and $g_I(x) \le w(x)$ for a.e. $x \in I$, where the constant is independent of I [36]. We call them basic weights because the weak type inequality

$$\int_{\{x:M^+f(x)>\lambda\}} w \le \frac{C}{\lambda} \int_{\mathbb{R}} |f|w$$

for weights $w \in A_1^+$ follows immediately (using the functions g_I) from the weak type inequality

$$\int_{\{x:M^+f(x)>\lambda\}} g \le \frac{1}{\lambda} \int_{\mathbb{R}} |f|g$$

for increasing weights g [36].

4.2 The doubling condition and examples of A_p^+ weights

If w is a weight in the Muckenhoupt A_p class, then it is a doubling weight, that is, there exists $C > 0$ such that

$$\int_{a-2h}^{a+2h} w \le C \int_{a-h}^{a+h} w$$

for all $a \in \mathbb{R}$ and $h > 0$. However, one-sided weights do not satisfy this property. For instance, $w(x) = \exp x$ is not a doubling weight, but it belongs to A_1^+ since it is increasing. The weights $w \in A_p^+$ satisfy a one-sided doubling condition: there exists $C > 0$ such that

$$\int_{a-h}^{a+h} w \le C \int_{a}^{a+h} w$$

for all $a \in \mathbb{R}$ and $h > 0$. Further, we notice that if $w \in A_p^+$ and w is a doubling weight, then $w \in A_p$.

We can give nontrivial examples of A_p^+ weights taking $w = gu$, where g is increasing and $v \in A_p$. We notice that there are other A_p^+ weights. For instance, take the function w defined as $w(x) = x^{1/2}$ in $(0, 1]$ and $w(x) = w(x+1)$ for all x. It is easy to see that w is in A_1^+ but w/g is not a doubling weight for any increasing function g and, therefore, w/g is not in A_1. We leave the details to the reader.

4.3 The reverse Hölder inequality

The usual proof of the strong type (p, p) inequality for Muckenhoupt A_p weights uses the idea that $w \in A_p$ implies that there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$. This property is a consequence of the fact that the Muckenhoupt weights satisfy the reverse Hölder inequality: if $w \in A_p$ then there exist $\delta > 0$ and $C > 0$ such that

$$\frac{1}{b-a}\int_a^b w^{1+\delta} \le \left(\frac{C}{b-a}\int_a^b w\right)^{1+\delta}$$

for all intervals (a, b).

Let us consider the same problem for the one-sided Hardy–Littlewood maximal operator M^+ and Sawyer's weights $w \in A_p^+$. If we look at the first proofs [51, 36] of the strong type inequality

$$\int |M^+ f|^p w \le C \int |f|^p w, \quad 1 < p < \infty,$$

for weights $w \in A_p^+$, we realize that they did not use the implication $w \in A_p^+ \Rightarrow w \in A_{p-\varepsilon}^+$. We notice that the weights in A_p^+ do not necessarily satisfy the reverse Hölder inequality. However, a substitute was found in [32]: if $w \in A_p^+$ then there exist positive constants C and δ such that for all a and b

$$\int_a^b w^{1+\delta} \le C(M^-(w\chi_{(a,b)})(b))^\delta \int_a^b w,$$

which implies

$$M^-(w^{1+\delta}\chi_{(a,b)})(b) \le C(M^-(w\chi_{(a,b)})(b))^{1+\delta}.$$

This is what we have called the weak reverse Hölder inequality (WRHI) since $(M^-(w\chi_{(a,b)})(b))^{1+\delta} \le M^-(w^{1+\delta}\chi_{(a,b)})(b)$ by the Hölder inequality. This condition together with $w \in A_p^+$ gives $w \in A_{p-\varepsilon}^+$ [32]. With this implication the proof of the strong type inequality of M^+ can follow the same steps as in the classical case.

We observe that this WRHI is not satisfactory in some sense because of the maximal operator M^- involved in it. We point out that it was proved in [32] (see also [37]) that the WRHI holds if and only if there exist positive numbers δ and C such that

$$\left(\frac{1}{c-a}\int_a^c w^{1+\delta}\right)^{\frac{1}{1+\delta}} \leq \frac{C}{b-a}\int_a^b w,$$

for all numbers $a < b$ and $c = (a+b)/2$, which seems to be a more natural formulation.

Once we know that the weights in A_p^+ satisfy WRHI the following questions remain open: It is known that the reverse Hölder inequality is equivalent to the fact that the weight is in some A_p class. Is this true for the WRHI and A_p^+ classes? Moreover, is there a concept of A_∞^+ weights, equivalent to the WRHI, analogous to the concept of A_∞ weights? The answers to these questions are affirmative.

Theorem 4.1 (*[37]) Let w be a nonnegative measurable function. The following statements are equivalent.*

(a) The weight w satisfies A_p^+ for some $p \geq 1$.
(b) The weight w satisfies the WRHI for some $\delta > 0$ and $C > 0$.
(c) w satisfies A_∞^+, i.e., there exist positive real numbers C and δ such that

$$\frac{\int\limits_E w\,d\mu}{\int\limits_a^c w\,d\mu} \leq C\left(\frac{|E|}{c-b}\right)^\delta$$

for all numbers $a < b < c$ and all subsets $E \subset (a,b)$.

Other proofs of this theorem can be found in [17] and [16].

4.4 Sharp functions and BMO

For a real locally integrable function f in the real line, the sharp maximal function $f^\sharp$ is defined at x by

$$f^\sharp(x) = \sup_{x\in I}\frac{1}{|I|}\int_I \left|f(y) - \frac{1}{|I|}\int_I f\right| dy$$

where the supremum is taken over all bounded intervals containing x, and $|I|$ stands for the Lebesgue measure of the interval I. If f is such that $f^\sharp \in L^\infty$, we say that f is a function of bounded mean oscillation and we write

$$BMO = \{f \in L^1_{loc} : f^\sharp \in L^\infty\}.$$

There is a close relation between BMO and A_p weights. More precisely, for fixed $p > 1$,

$$BMO = \{\alpha \log w : w \in A_p, \alpha \geq 0\}.$$

In the one-sided setting [39], the one-sided sharp maximal function is defined as

$$f^\sharp_+(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy,$$

where $z^+ = \max(z, 0)$. ($f^\sharp_-$ is defined in the obvious way.) We say that a function f is in BMO^+ if $f^\sharp_+ \in L^\infty$ and we write

$$\|f\|_{*,+} = \|f^\sharp_+\|_\infty.$$

Analogously, f is in BMO^- if $f^\sharp_- \in L^\infty$ and

$$\|f\|_{*,-} = \|f^\sharp_-\|_\infty.$$

It is clear that $\|\cdot\|_{*,+}$ and $\|\cdot\|_{*,-}$ are not norms and $BMO = BMO^+ \cap BMO^-$.

BMO^+ functions satisfy a kind of John–Nirenberg inequality: If $f \in BMO^+$ then there exist positive constants K and α such that for every interval I and all $\lambda > 0$

$$|\{x \in I^- : (f(x) - \frac{1}{|I^+|} \int_{I^+} f)^+ > \lambda\}| \leq K|I| \exp\left(\frac{-\alpha\lambda}{\|f\|_{*,+}}\right),$$

where $|I^-| = |I^+| = |I|$ and are contiguous to I to the left and right, respectively. BMO^+ is related to A^+_p weights in the same way that BMO and A_p are related, i.e.,

$$BMO^+ = \{\alpha \log w : w \in A^+_p, \alpha \geq 0\} \quad (p > 1).$$

The maximal functions M^+ and $f^\sharp_+$ are also related, as the following theorem shows.

Theorem 4.2 ([37]) *Assume $w \in A^+_\infty$, $f \geq 0$ and $M^+f \in L^{p_0}(w)$ for some p_0, $0 < p_0 < \infty$. Then for every p, $p_0 \leq p < \infty$,*

$$\int_{-\infty}^{\infty} (M^+f)^p w \leq C \int_{-\infty}^{\infty} {f^\sharp_+}^p w.$$

This result is very useful for studying the weights for the one-sided fractional integral, as we show below.

Let $0 < \alpha < 1$. The one-sided fractional integral and the one-sided fractional maximal operator are defined by

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\,dy \qquad \text{and} \qquad N_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(y)|\,dy.$$

Notice that $N_\alpha^+ \le I_\alpha^+$. The weights for these operators have been studied in [4] (see also [37] and [39]). In particular, the problem of characterizing the weights $w > 0$ such that

$$\left(\int_{\mathbb{R}} |I_\alpha^+ f|^q w^q\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}} |f|^p w^p\right)^{\frac{1}{p}}$$

with $\frac{1}{q} = \frac{1}{p} - \alpha$, $1 < p < \frac{1}{\alpha}$, and $0 < \alpha < 1$ has been solved in [4] and [37] by reducing it to the corresponding problem for N_α^+, i.e.,

$$\left(\int_{\mathbb{R}} |N_\alpha^+ f|^q w^q\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}} |f|^p w^p\right)^{\frac{1}{p}}.$$

In [4] the reduction is done by using an argument of Welland, while in [37] it is proved using a distribution function weighted inequality for A_∞^+ weights which yields as a corollary the inequality

$$\left(\int_{\mathbb{R}} |I_\alpha^+ f|^q w^q\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}} |N_\alpha^+ f|^q w^q\right)^{\frac{1}{q}}. \tag{6}$$

We want to point out here that the one-sided sharp function together with Theorem 4.2 gives easily the same inequality (6). This is the approach in [39]. The key fact is in the following lemma.

Lemma 4.1 *If* $0 < \alpha < 1$ *and* f *is a nonnegative measurable function, then*

$$(I_\alpha^+ f)_+^\sharp \le C_\alpha N_\alpha^+ f.$$

We sketch the proof. Let us fix $h > 0$ and x. Let $f_1 = f\chi_{(x,x+2h)}$ and $f_2 = f - f_1$. Then

$$\begin{aligned} &\tfrac{1}{h}\int_x^{x+h} \left(I_\alpha^+ f(y) - \tfrac{1}{h}\int_{x+h}^{x+2h} I_\alpha^+ f(z)dz\right)^+ dy \\ \le\ & \tfrac{1}{h}\int_x^{x+h} \left(I_\alpha^+ f_1(y) - \tfrac{1}{h}\int_{x+h}^{x+2h} I_\alpha^+ f_1(z)dz\right)^+ dy \\ +\ & \tfrac{1}{h}\int_x^{x+h} \left(I_\alpha^+ f_2(y) - \tfrac{1}{h}\int_{x+h}^{x+2h} I_\alpha^+ f_2(z)dz\right)^+ dy. \end{aligned}$$

The first term on the right-hand side is obviously bounded by

$$\frac{1}{h}\int_x^{x+2h} I_\alpha^+ f_1(y)dy \le \frac{1}{h}\int_x^{x+2h}\int_y^{x+2h} \frac{f(z)}{(z-y)^{1-\alpha}}dzdy \le C_\alpha N_\alpha^+ f(x).$$

The second term on the right-hand side is zero because the term inside the brackets is nonpositive. These two estimates prove the lemma.

Now, it is clear that the lemma and Theorem 4.2 give inequality (6).

4.5 The discrete version

Everything we have said about the one-sided weights admits a discrete version for functions defined on $\mathbb{Z}$. In particular, given a weight on $\mathbb{Z}$, if f is a function defined on $\mathbb{Z}$ and

$$m^+ f(j) = \sup_{n\in\mathbb{N}} \frac{1}{n+1} \sum_{i=0}^{n} |f(i+j)|,$$

then m^+ is bounded on $l^p(w)$, $1 < p < \infty$, if and only if w satisfies A_p^+ on the integers, i.e., there exists a constant C such that

$$\left(\frac{1}{r-m+1} \sum_{i=m}^{n} w(i) \right)^{1/p} \left(\frac{1}{r-m+1} \sum_{i=n}^{r} w^{1-p'}(i) \right)^{1/p'} \le C$$

for all integers $m \le n \le r$. The natural weak type characterization holds for $p = 1$.

5 Some approximations of the identity

Let φ be a nonnegative integrable function in $\mathbb{R}$. For each $R > 0$ let $\varphi_R(x) = \frac{1}{R}\varphi(\frac{x}{R})$. It is well known that for all $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, the convolutions $f * \varphi_R$ converge in $L^p(\mathbb{R})$ to $(\int \varphi) f$ as R goes to zero. The study of the a.e. convergence of $f * \varphi_R$ is harder, and we need to add certain assumptions on φ. If φ is radially decreasing, then the maximal operator associated to φ

$$M_\varphi f(x) = \sup_{R>0} |f * \varphi_R(x)|,$$

satisfies the inequality

$$M_\varphi f(x) \le \left(\int_{\mathbb{R}} \varphi \right) M f(x),$$

where M is the Hardy–Littlewood maximal operator. Therefore, M_φ behaves as M and $f * \varphi_R$ converges a.e. for all $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, as R goes to zero. Similarly, if $\varphi : \mathbb{R} \to \mathbb{R}$ is integrable, decreasing in $(0, \infty)$ and $\varphi(x) = 0$ in $(-\infty, 0)$, then

$$M_\varphi f(x) \le \left(\int_{\mathbb{R}} \varphi \right) M^- f(x).$$

Therefore, the weighted inequalities for M_φ follow from the corresponding ones for M^-. But, what can be said if φ is a general integrable function? An interesting particular example is the Cesàro maximal operator defined by

$$M_\alpha^+ f(x) = \sup_{R>0} \frac{1}{R} \int_x^{x+R} |f(y)| \left(\frac{x+R-y}{R} \right)^\alpha dy, \quad -1 < \alpha < 0.$$

This operator appears in the study of the Cesàro continuity [28]. Observe that M_α^+ is the operator M_ψ for $\psi(x) = (1+x)^\alpha \chi_{(-1,0)}(x)$ and $M_\alpha^+ = M^+$ when $\alpha = 0$. If $p > 1/(1+\alpha)$ then $\psi \in L^{p'}(dx)$ and Hölder's inequality gives that

$$M_\alpha^+ f \le C[M(|f|^p)]^{1/p}.$$

It follows that M_α^+ is of strong type (p,p) for $p > 1/(1+\alpha)$. For $p = 1/(1+\alpha)$ (which is strictly greater than 1 if $\alpha < 0$) we have that M_α^+ is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$ [28], i.e., there exists $C > 0$ such that

$$|\{x \in \mathbb{R} : M_\alpha^+ \chi_E(x) > \lambda\}| \le \frac{C}{\lambda^{1/(1+\alpha)}} |E|$$

for all measurable sets E. The proof follows from the inequality

$$M_\alpha^+ \chi_E \le C[M^+(\chi_E)]^{1+\alpha},$$

which is obtained using Hölder's inequality for $L^{p,q}$-spaces (it cannot be obtained using Hölder's inequality for Lebesgue spaces since $\psi \notin L^{-1/\alpha}(dx)$).

The good weights for M_α^+ have been studied in [40]. Since M_α^+ is a one-sided operator, is not surprising that one-sided weights appear. The main result in [40] says that M_α^+ applies $L^p(w)$ into $L^p(w)$ if and only if $w \in A_{p,\alpha}^+$, $1 < p < \infty$, i.e., there exists $C > 0$ such that

$$\left(\int_a^b w(s)\, ds\right) \left(\int_b^c w^{1-p'}(s)(c-s)^{\alpha p'}\, ds\right)^{p-1} \le C(c-a)^{p(1+\alpha)}$$

for all numbers $a < b < c$. These classes are related to Sawyer's classes in the following way: If $p > 1/(1+\alpha)$ then

$$A_{p(1+\alpha)}^+ \subset A_{p,\alpha}^+ \subsetneqq A_p^+$$

and we do not know whether $A_{p(1+\alpha)}^+ = A_{p,\alpha}^+$. Restricted weak type weighted inequalities are also characterized in [40]. For instance, M_α^+ is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$ with respect to the measure $w(x)\, dx$, i.e.,

$$\int_{\{x \in R : M_\alpha^+ \chi_E(x) > \lambda\}} w \le \frac{C}{\lambda^{1/(1+\alpha)}} \int_E w,$$

if and only if $w \in A_1^+$.

As we said above, $M_\alpha^+ = M_\psi$ for $\psi(x) = (1+x)^\alpha \chi_{(-1,0)}(x)$. Notice that ψ is a translation of a decreasing function in $(0,\infty)$ with support in $(0,\infty)$. More precisely, $\psi(x) = \varphi(x+1)$, where $\varphi(x) = x^\alpha \chi_{(0,1)}(x)$. If we take $\widetilde{\psi}(x) = \varphi(x-1)$, then $M_{\widetilde{\psi}}$ is the operator

$$\widetilde{M}_\alpha^- f(x) = \sup_{R>0} \frac{1}{R} \int_{x-2R}^{x-R} |f(y)| \left(\frac{x-R-y}{R} \right)^\alpha dy \quad .$$

Just as M_α^+ is related to the Cesàro continuity [28], the operator $\widetilde{M}_\alpha^-$ is related to the Cesàro convergence of singular integrals [7]. It is known that $\widetilde{M}_\alpha^-$ is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$, and of strong type (p,p) for $p > 1/(1+\alpha)$ and it is not of weak type $(1/(1+\alpha), 1/(1+\alpha))$. Weighted inequalities for $\widetilde{M}_\alpha^+$ have been studied in [8]. We have that M_α^+ and $\widetilde{M}_\alpha^-$ have essentially the same behavior and they are particular cases of the maximal operator

$$M_{\tau_h\varphi} f(x) = \sup_{R>0} |f| * [\tau_h\varphi]_R(x),$$

where $h \neq 0$, $\tau_h\varphi(x) = \varphi(x-h)$, and φ is a decreasing function in $(0,\infty)$ with support in $(0,\infty)$. It is then interesting to study the boundedness of $M_{\tau_h\varphi}$ for general φ. We observe that the support of $\tau_h\varphi$ is not necessarily contained in $(0,\infty)$; if so, then $\tau_h\varphi$ is not necessarily bounded for a decreasing function in $(0,\infty)$. It follows that one cannot apply the classical theory to study the boundedness of $M_{\tau_h\varphi}$ and, consequently, the a.e. convergence of $f * [\tau_h\varphi]_R$ (however, the convolutions $f * [\tau_h\varphi]_R$ converge in $L^p(\mathbb{R})$, $1 \leq p < \infty$, since $\tau_h\varphi$ is integrable). Then the following question arises: Is the behavior of the maximal operator $M_{\tau_h\varphi}$ with respect to the Lebesgue measure analogous to that of $\widetilde{M}_\alpha^-$ and M_α^+? More precisely, is it always true that for all φ there exists $p_0 \geq 1$ such that $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p > p_0$ and $M_{\tau_h\varphi}$ is of restricted weak type (p_0,p_0)? It was seen in [10] that the answer is negative. In fact, the following situations are possible for $p_0 \geq 1$:

(i) $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p > p_0$ but $M_{\tau_h\varphi}$ is not of restricted weak type (p_0,p_0).
(ii) $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p > p_0$ but $M_{\tau_h\varphi}$ is of restricted weak type (p,p) if and only if $p \geq p_0$ (the case of $\widetilde{M}_\alpha^-$ and M_α^+).
(iii) $M_{\tau_h\varphi}$ is of weak type (p,p) if and only if $p \geq p_0$ but $M_{\tau_h\varphi}$ is not of restricted weak type (p,p) if $p < p_0$.

Once we know that, it is interesting to study weak type inequalities and restricted weak type inequalities for $M_{\tau_h\varphi}$ in weighted L^p-spaces. This was done in [9] and [10], where the characterizations of these weighted inequalities can be found.

6 One-sided singular integrals

The two-sided Hardy–Littlewood maximal operator plays a key role in the study of singular integrals. It is natural to ask whether there exist singular integrals which are controlled by the one-sided Hardy–Littlewood maximal operator M^+. As we will see below, Carlos Segovia found a natural candidate.

6.1 One-sided strongly singular integrals

In the beginning of the 1990s C. Segovia (with J. Garcia-Cuerva, E. Harboure, and J.L. Torrea) studied weighted inequalities for commutators of strongly singular integrals [22]. A classical example of a kernel of this type of singular integral in dimension 1 is $\frac{\exp(i|x|^{-1})}{|x|}$. These singular integrals were studied by many authors. The weak type $(1, 1)$ was obtained by C. Fefferman [19] in 1970. Fourteen years later, S. Chanillo in [14] generalized this result for weights in the Muckenhoupt class A_1. To obtain this result, Chanillo used, among other things, a theorem of complex interpolation between weighted Hardy spaces (for weights in the class A_p of Muckenhoupt). This interpolation theorem is also contained essentially in a more general result due to Strömberg and Torchinsky [58]. In those years C. Segovia became interested in the theory of one-sided weights. Segovia realized that, unlike the case of the general Calderón–Zygmund kernels (you can think of the Hilbert kernel $k(x) = \frac{1}{x}$), we can truncate the strongly singular kernel and still have a kernel with the suitable cancellation property. In this way a natural one-sided singular kernel is

$$\frac{\exp(i|x|^{-1})}{|x|}\chi_{(-\infty,0)}(x). \tag{7}$$

Then a natural question is the following: For the operator associated with the kernel (7), is it possible to obtain the weak type $(1, 1)$ for a weight $w \in A_1^+$? The strong singularity of this kernel does not allow us to use standard techniques to answer this question, and the unique antecedent associated with the problem was Chanillo's result. But in that moment there was no theory of weighted Hardy spaces for weights in the class A_p^+. This was the main motivation for Segovia and L. de Rosa to develop a great part of the theory of one-sided Hardy spaces in three successive papers [52], [53], [54]. Furthermore, recently in [45] an interpolation result has been proved between one-sided Hardy spaces, that generalizes the one obtained by Strömberg and Torchinsky. Finally, C. Segovia and R. Testoni in [55] proved the weak type $(1, 1)$, with respect to a weight $w \in A_1^+$, of the operator associated with the kernel (7), in part, using the interpolation result that appears in [45]. They also proved another interesting application of this result related to a theorem of multipliers in one-sided Hardy spaces (see [56]). We notice that Segovia and Testoni [55] have also studied the strong type (p, p), $1 < p < \infty$, of the operator associated with the kernel (7). In this case, the result can be obtained following Chanillo's proof replacing the sharp function $f^\sharp$ by the one-sided version $f_+^\sharp$. However, in the case $p = 1$ they follow Chanillo's argument as far as they can, but they need a different technique which uses the weighted results for the one-sided fractional integral. They point out that this new argument is simpler even for weights in the Muckenhoupt classes.

6.2 One-sided Calderón–Zygmund kernels

We say that a function k in $L^1_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\left| \int_{\varepsilon<|x|<N} k(x)\, dx \right| \le B_1 \qquad \text{for all } \varepsilon \text{ and all } N, \text{ with } 0<\varepsilon<N,$$

and furthermore $\lim_{\varepsilon \to 0^+} \int_{\varepsilon<|x|<1} k(x)\, dx$ exists,

(b) there exists a finite constant B_2 such that

$$|k(x)| \le \frac{B_2}{|x|}, \quad \text{for all } x \neq 0,$$

(c) there exists a finite constant B_3 such that

$$|k(x-y) - k(x)| \le B_3 |y| |x|^{-2} \quad \text{for all } x \text{ and } y \text{ with } |x| > 2|y| > 0.$$

Associated to k we consider the maximal operator

$$T^* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} k(x-y) f(y)\, dy \right|$$

and the singular integral

$$Tf(x) = P.V. \int k(x-y) f(y)\, dy = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} k(x-y) f(y)\, dy.$$

It is a well known result (see [15] and [23]) that if w satisfies the Muckenhoupt A_p condition and $1<p<\infty$, then

$$\begin{aligned} \int_{-\infty}^{\infty} |Tf(x)|^p w(x)\, dx &\le \int_{-\infty}^{\infty} |T^* f(x)|^p w(x)\, dx \\ &\le C \int_{-\infty}^{\infty} |f(x)|^p w(x)\, dx, \end{aligned}$$

while if $w \in A_1$ then

$$w(\{|Tf(x)| > \lambda\}) \le w(\{T^* f(x) > \lambda\}) \le \frac{C}{\lambda} \int_{-\infty}^{\infty} |f(x)| w(x)\, dx,$$

where the constants C are independent of f and λ. If we consider the Hilbert transform

$$Hf(x) = P.V. \int \frac{f(y)}{x-y}\, dy,$$

i.e., the singular integral associated to the kernel $k(x) = 1/x$, then condition A_p is necessary for the above inequalities to hold [26].

If there are singular integrals associated to Calderón–Zygmund kernels which are controlled by M^+, then the support of the kernels should be contained in $(-\infty, 0)$ and still keep the cancellation property (a). This means that we cannot take $k(x) = \frac{1}{x}\chi_{(-\infty,0)}$. However, the class of general Calderón–Zygmund kernels supported in a half-line is nontrivial. For instance,

$$k(x) = \frac{1}{x} \cdot \frac{\sin(\log|x|)}{\log|x|} \cdot \chi_{(-\infty,0)}(x)$$

is a Calderón–Zygmund kernel. It turns out that A_p^+ weights are good weights for the singular integral associated to a Calderón–Zygmund kernel with support in $(-\infty, 0)$.

Theorem 6.1 (*[1]) Let k be a Calderón–Zygmund kernel with support in $\mathbb{R}^- = (-\infty, 0)$.*

(a) If $w \in A_p^+$ with $1 < p < \infty$, then there exists a constant C depending only on B_1, B_2, B_3, p, and the constant in the condition A_p^+, such that

$$\int_{-\infty}^{\infty} |T^* f(x)|^p w(x)\, dx \le C \int_{-\infty}^{\infty} |f(x)|^p w(x)\, dx \ ,$$

for all $f \in L^p(w)$,

(b) If $w \in A_1^+$ then there exists a constant C depending only on B_1, B_2, B_3, and the constant in the condition A_1^+ such that

$$w(\{T^* f(x) > \lambda\}) \le \frac{C}{\lambda} \int_{-\infty}^{\infty} |f(x)| w(x)\, dx$$

for all $f \in L^1(w)$ and all $\lambda > 0$.

The proof uses distribution function inequalities and the theory of A_∞^+ weights [37]. Alternatively, it is possible to give a proof based on the one-sided sharp maximal function. An analogous result holds for A_p^- weights, $1 \le p \le \infty$, and singular integrals associated to Calderón–Zygmund kernels with support in $(0, \infty)$.

Consider for any $\lambda > 0$ the dilation of the kernel k given by

$$k_\lambda(x) = \lambda k(\lambda x) \ .$$

It is clear that if k is a Calderón–Zygmund kernel, then so is k_λ and with the same constants B_1, B_2, and B_3 as k. If T_λ^* are the maximal singular integrals associated to the dilations k_λ, then T_λ^* are uniformly bounded from $L^p(w)$ into $L^p(w)$ if w satisfies A_p^+, $1 < p < \infty$, and from $L^1(w)$ into weak-$L^1(w)$ if w satisfies A_1^+. The next theorem is a kind of converse of this remark and includes a two-sided version.

Theorem 6.2 *([1]) Let k be a Calderón–Zygmund kernel. For each $\lambda > 0$ let T_λ^* denote the maximal operator with kernel k_λ and let $1 \le p < \infty$. Let w be a positive measurable function and assume that the operators T_λ^* are uniformly bounded from $L^p(w)$ into weak-$L^p(w)$.*

(a) If there exists $x_0 < 0$ such that $k(x_0) \neq 0$, then $w \in A_p^+$.
(b) If there exists $x_1 > 0$ such that $k(x_1) \neq 0$, then $w \in A_p^-$.
(c) If there exist $x_0 < 0 < x_1$ such that $k(x_0) \neq 0 \neq k(x_1)$, then $w \in A_p$.

Theorems 6.1 and 6.2 also hold for the singular integral

$$Tf(x) = \lim_{\varepsilon \to 0^+} T_\varepsilon f(x) \ .$$

The proofs for T are similar to the corresponding ones for T^* or follow easily from the theorem for T^*.

Since the Hilbert kernel $k(x) = 1/x$ coincides with its dilations, we have that statement (c) of Theorem 6.2 gives the necessary part of the theorem by Hunt, Muckenhoupt, and Wheeden [26] which characterizes the good weights for the Hilbert transform.

6.3 Further examples of one-sided singular kernels

Some singular integrals with kernel supported in $(-\infty, 0)$ appear in the study of the convergence of the ergodic averages associated to a measurable flow or to a measure-preserving transformation [27], [5]. We are going to state the problem in the real line which is enough to see how these kernels appear. The ergodic results follow from the real line case [6].

If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and

$$D_k f(x) = \frac{1}{2^k} \int_0^{2^k} f(x+t)\, dt, \quad k \in \mathbb{Z}, \tag{8}$$

then $\lim_{k\to-\infty} D_k f(x) = f(x)$ and $\lim_{k\to\infty} D_k f(x) = 0$ a.e. In order to give some information about how the convergence occurs, we may consider the series

$$\sum_{k=-\infty}^{\infty} (D_k f(x) - D_{k-1} f(x)),$$

which obviously converges a.e. It is natural to ask about the convergence properties of

$$\sum_{k=-\infty}^{\infty} v_k (D_k f(x) - D_{k-1} f(x)), \tag{9}$$

where v_k is a bounded sequence of real or complex numbers. In other words, we wish to study the convergence of the partial sums

$$T_N f(x) = \sum_{k=N_1}^{N_2} v_k(D_k f(x) - D_{k-1} f(x)), \quad N = (N_1, N_2),$$

as $N_1 \to -\infty$ and $N_2 \to \infty$. We observe that T_N is a convolution operator since $T_N f(x) = K_N * f(x)$, where

$$K_N(x) = \sum_{k=N_1}^{N_2} v_k(\varphi_k(x) - \varphi_{k-1}(x)), \quad \varphi_k(x) = \frac{1}{2^k}\chi_{(-2^k,0)}(x).$$

In order to prove the a.e. convergence, we consider the maximal operator

$$T^* f(x) = \sup_N |T_N f(x)|.$$

First of all we obtain uniform boundedness of the operators T_N. The kernels K_N have support in $(-\infty, 0)$, but they are not Calderón-Zygmund kernels (in the above sense). However, an easy computation shows that

$$\sup_N |\widehat{K_N}(x)| \leq C,$$

where $\widehat{K_N}$ stands for the Fourier transform of the kernel. It follows that the operators T_N are uniformly bounded in $L^2(dx)$, i.e.,

$$\sup_N \|T_N f\|_{L^2(dx)} \leq C\|f\|_{L^2(dx)}.$$

Another property, less evident, is the condition D_r, $1 \leq r < \infty$: If $1 \leq r < \infty$ and $2^{i-1} < x \leq 2^i$, then

$$\sum_{j=i+1}^{\infty} 2^{j/r'} \left(\int_{2^j}^{2^{j+1}} |K_N(x-y) - K_N(-y)|^r \, dy \right)^{1/r} < \infty.$$

Condition D_1 implies Hörmander's condition:

$$\int_{\{y:|y|>C|x|\}} |K_N(x-y) - K_N(-y)| \, dy \leq C,$$

which together with the uniform boundedness of the Fourier transform gives (see [23])

$$\sup_N |\{x : |T_N f(x)| > \lambda\}| \leq \frac{C}{\lambda} \int |f|$$

and

$$\sup_N \int |T_N f|^p \leq C_p \int |f|^p, \quad 1 < p < \infty.$$

To study the uniform boundedness of T_N in weighted spaces, we use the one-sided sharp maximal function. It follows from the uniform boundedness of T_N and condition D_r that for every $s > 1$ there exists C such that

$$[|T_N f|]_+^{\#}(x) \le C\,(M^+|f|^s)^{1/s}(x).$$

This inequality and the relation between M^+f and $f_+^{\#}$ allow us to get the uniform boundedness of T_N in weighted spaces.

Theorem 6.3 (*[5]*) *Let* $1 < p < \infty$ *and* $w \in A_p^+$. *There exists* C *such that*

$$\int |T_N f|^p w \le C \int |f|^p w \;.$$

Proof We sketch the proof of the theorem. Since $w \in A_p^+$, there exists $s > 1$ such that $w \in A_{p/s}^+$ [51]. Therefore,

$$\int |T_N f|^p w \le \int |M^+(T_N f)|^p w \le C \int (|T_N f|_+^{\#})^p w.$$

Now, the last term is dominated by

$$C \int (M^+|f|^s)^{p/s} w.$$

The proof is completed by using the fact that $w \in A_{p/s}^+$ and the characterization of the strong type inequality for M^+.

We notice that if we use the classical sharp maximal function, then we obtain the last result only for weights in the Muckenhoupt classes which are a subclass of A_p^+ (we recall that the increasing weights belong to the A_p^+ classes).

In order to get the boundedness of the maximal operator, the key result is the following inequality: If

$$T_M^* f(x) = \sup_{|N_1|,|N_2| \le M} |T_{(N_1,N_2)} f(x)|,$$

then for every $s \in (1,\infty)$ there exists a constant C such that

$$T_M^* f(x) \le C \left[M^+(|T_{-M,M} f|)(x) + \left(M^+|f|^s\right)^{1/s}(x) \right].$$

Combining this inequality and the uniform boundedness of the operators T_N in weighted spaces, we easily obtain the following theorem.

Theorem 6.4 (*[5]*) *If* $1 < p < \infty$ *and* $w \in A_p^+$, *then there exists* C *such that*

$$\int_{\mathbb{R}} |T^* f|^p w \le C \int_{\mathbb{R}} |f|^p w, \quad f \in L^p(w),$$

and $T_N f$ *converges a.e. and in* $L^p(w)$ *for all* $f \in L^p(w)$.

The result about the convergence follows from the strong type inequality for T^* and the convergence for the functions in the Schwartz class.

If $p = 1$ we obtain the following.

Theorem 6.5 ([5]) *Let* $w \in A_1^+$. *Then*

$$\int_{\{x:T^*f(x)>\lambda\}} w \leq \frac{C}{\lambda}\int_{\mathbb{R}} |f|w \quad \text{for all } \lambda > 0 \text{ and } f \in L^1(w).$$

Further, $T_N f$ *converges a.e. and in measure for all* $f \in L^1(w)$.

If we work in a vector-valued setting, we obtain results of boundedness and convergence for more general operators. For instance, we can prove that if $w \in A_p^+$, $1 < p < \infty$, then the square function

$$Sf(x) = \left(\sum_{k=-\infty}^{\infty} |D_k f(x) - D_{k-1}f(x)|^2\right)^{1/2}$$

applies $L^p(w)$ into $L^p(w)$. This result was previously obtained in [61].

The same problem can be studied for the Cesàro α averages, $-1 < \alpha \leq 0$, defined by

$$D_k f(x) = \frac{1+\alpha}{2^{k(1+\alpha)}}\int_x^{x+2^k} (x + 2^k - t)^\alpha f(t)\,dt. \tag{10}$$

We have $D_k f(x) = \varphi_k * f(x)$, where $\varphi_k(x) = \frac{1}{2^k}\varphi(x/2^k)$ and $\varphi(x) = (1 + \alpha)(1 + x)^\alpha \chi_{(-1,0)}(x)$. Now the operators $T_N = T_N^\alpha$ are convolution operators with kernels K_N^α supported in $(-\infty, 0)$, where

$$K_N^\alpha(x) = \sum_{k=N_1}^{N_2} v_k[\varphi_k(x) - \varphi_{k-1}(x)].$$

As before, the Fourier transforms of the kernels are uniformly bounded, that is, there exists a constant C depending only on α and $\|v_k\|_\infty$ such that

$$\sup_N |\widehat{K_N^\alpha}(\xi)| \leq C,$$

for all $\xi \in \mathbb{R}$. However, the kernels K_N^α satisfy condition D_r only in the range $1 \leq r < -1/\alpha$. These properties give the uniform boundedness of the operators $T_N = T_N^\alpha$ in $L^p(dx)$, but the estimate

$$[|T_N^\alpha f|]_+^\#(x) \leq C\, M_s^+ f(x)$$

holds only for $s > \frac{1}{1+\alpha}$ (observe that $\frac{1}{1+\alpha}$ is the conjugate exponent of $-1/\alpha$). Using the same ideas as in the case $\alpha = 0$, we obtain the following theorems for the maximal operators

$$T^* f = T^*_\alpha f = \sup_N |T^\alpha_N f|.$$

Theorem 6.6 *Let $\{v_k\}$ be a multiplying sequence and $-1 < \alpha \leq 0$. If $1/(1+\alpha) < p < \infty$ and $w \in A^+_{p(1+\alpha)}$, then there exists a constant C such that*

$$\|T^* f\|_{L^p(w)} \leq C \, \|f\|_{L^p(w)}$$

for all functions $f \in L^p(w)$.

We remark that the restriction $\frac{1}{1+\alpha} < p$ follows from the fact that the kernels satisfy D_r only in the range $1 \leq r < -1/\alpha$.

Theorem 6.7 *Let $\{v_k\}$ be a multiplying sequence and $-1 < \alpha \leq 0$. If $w^{1/(1+\alpha)} \in A^+_1$ then there exists a constant C such that*

$$\int_{\{x \in \mathbb{R}:\ |T^* f(x)| > \lambda\}} w \leq \frac{C}{\lambda} ||f||_{L^1(w)},$$

for all $\lambda > 0$ and all functions $f \in L^1(w)$.

Using these theorems and proving the a.e. convergence in the Schwartz class, we obtain that the operators $T^\alpha_N f$ converge a.e. in the suitable spaces. We remark that these theorems have been obtained in [11] and they are new even in the unweighted case ($w \equiv 1$).

7 Some applications to ergodic theory

We started this paper showing how a classical ergodic theorem gave information about one-sided weights. In this section we will see how the theory of weights for one-sided operators has important applications to ergodic theory.

Let (X, Σ, μ) be a sigma-finite measure space and $T : X \to X$ a measure-preserving transformation. A central problem in ergodic theory is the a.e. convergence of the averages $A_n(T)f(x) = \frac{1}{n+1} \sum_{i=0}^n f(T^i x)$ for every f in some $L^p(X)$. It is well known that the a.e. convergence for all $f \in L^p(X)$ follows if we have it for every f in a dense class and we control the maximal operator

$$\mathcal{M} f(x) = \sup_{n \geq 0} |A_n(T) f(x)| \ .$$

One easy example is obtained by taking $X = \mathbb{Z}$, with counting measure and $T(n) = n + 1$. In this case, the ergodic maximal function is the Hardy–Littlewood maximal operator

$$m^+ f(j) = \sup_{n \geq 0} \frac{1}{n+1} \left| \sum_{i=0}^n f(j+i) \right|,$$

which is clearly a one-sided operator. A famous theorem of A. Calderón [13] says that this is the only interesting example! Indeed, Calderón proved that from the fact that m^+ is a bounded operator in $L^p(\mathbb{Z})$, $p > 1$, and of weak type $(1,1)$, it follows that the same holds for $\mathcal{M}$ in $L^p(X)$. A natural question is: Can we get a theorem in the spirit of Calderón's for maximal operators associated to a wider class than measure-preserving transformations?

7.1 The strong type

Let us assume that T, instead of being a point transformation, is a linear operator acting on some $L^p(X)$. In this case, as before, we may define the ergodic maximal function associated to T as

$$\mathcal{M}(T)f(x) = \sup_{n\geq 0} |A_n(T)f(x)| \ ,$$

where $A_n(T)f(x) = \frac{1}{n+1}\sum_{i=0}^{n} T^i f(x)$. Is it possible to obtain the boundedness of $\mathcal{M}$ from some result on the integers? The answer is yes. In order to understand how to pass from $\mathbb{Z}$ to X, let us have a closer look at the proof of Calderón. It has four main ingredients.

(1) The operator $\mathcal{M}$ is the monotone limit of operators

$$\mathcal{M}_N f(x) = \sup_{N>n\geq 0} |A_n(T)f(x)| \ .$$

(2) The operators $\mathcal{M}_N$ commute with T.
(3) There are functions $h_i(x)$ such that

$$\int_X \mathcal{M}_N f(x)^p h_i(x)\, d\mu(x) = \int_X T^i(\mathcal{M}_N f)^p(x)\, d\mu(x)$$

(namely $h_i(x) = 1$).
(4) A nice theorem in the integers.

Let us see how they are put to work.

Theorem 7.1 *Let $1 < p < \infty$. Let T be an operator defined on $L^p(X)$. Let S be an operator induced in some way by T and let us assume that there are operators S_N such that $Sf(x)$ is the monotone limit of the $S_N f(x)$ and such that:*

(a) T^j commutes with S_N for every j and N,
(b) there exist functions $h_j(x)$ such that

$$\int_X |T^j f(x)|^p h_j(x) d\mu(x) = \int |f(x)|^p d\mu(x),$$

(c) there are operators R_N acting on ℓ_p such that

$$S_N(T^j f)(x) \le C R_N f_x(j) \ ,$$

where $f_x(j) = T^j f(x)$.

(d) there exists C such that for any N, L and a.e. x

$$\sum_{j=0}^{L} |R_N g(j)|^p h_j(x) \le C \sum_{j=0}^{L+N} |g(j)|^p h_j(x) \ .$$

Then there exists C such that

$$\int_X |Sf(x)|^p \, d\mu(x) \le C \int_X |f|^p \, d\mu(x) \ .$$

Proof. Let $f_x(j) = T^j f(x)$. Then

$$\begin{aligned}
\int_X |S_N f(x)|^p \, d\mu(x) &= \frac{1}{L+1} \sum_{j=0}^{L} \int_X |T^j(S_N f)(x)|^p h_j(x) \, d\mu(x) \\
&= \frac{1}{L+1} \sum_{j=0}^{L} \int_X |S_N(T^j f)(x)|^p h_j(x) \, d\mu(x) \\
&\le \frac{C}{L+1} \sum_{j=0}^{L} \int_X |R_N f_x(j)|^p h_j(x) \, d\mu(x) \\
&\le \frac{C}{L+1} \int_X \sum_{j=0}^{L+N} |f_x(j)|^p h_j(x) \, d\mu(x) \\
&= C \frac{L+N+1}{L+1} \int_X |f|^p \, d\mu(x).
\end{aligned}$$

Letting L and then N go to infinity we are done.

Examples of operators satisfying (b) are the Lamperti operators, which include the class of positive, with positive inverse, linear operators in $L^p(X)$ [29]. The following theorem is an example of the power of the transference method.

Theorem 7.2 (*[38]) Let $1 < p < \infty$. Let $T : L^p(X) \to L^p(X)$ be a positive, with positive inverse, linear operator. Then*

$$\int_X \mathcal{M} f(x)^p \, d\mu \le C \int_X |f(x)|^p \, d\mu$$

if, and only if, there exists C such that

$$\sup_n \int_X |A_n f(x)|^p \, d\mu \le C \int_X |f(x)|^p \, d\mu.$$

Since the only if part is trivial, we only need to check that the conditions of Theorem 7.1 are satisfied. The existence of the h_j was a known fact [29]. The hardest part is to show that the functions $h_i(x)$ satisfy condition A_p^+ on the integers with a constant independent of x. This is done using a modification of the Rubio de Francia algorithm [47]. The last hypothesis of 7.1 is now a consequence of the theory of weights for the one-sided Hardy–Littlewood maximal operator. In a similar way one can prove L^p-boundedness of many operators associated to mean bounded linear transformations.

7.2 The weak type

Let us analyze the problem of transferring weak type inequalities. Let O_λ^N be the set of x such that $S_N f(x) > \lambda$. The first step is as before.

$$\mu(O_\lambda^N) = \int_X \chi_{O_\lambda^N}\, d\mu(x) = \frac{1}{L+1}\sum_{j=0}^{L}\int_X T^j \chi_{O_\lambda^N} h_j(x)\, d\mu(x).$$

If we want to use the weak type of the operator R_N in the integers, we need to relate $T^j\chi_{O_\lambda^N}$ with the characteristic function of the set of j such that $R_N f_x(j) > \lambda$. We would need, for example, $T^j\chi_{O_\lambda^N} = \chi_{\{x:R_N T^j f(x)>\lambda\}}$. This is not to be expected unless T maps characteristic functions into characteristic functions, which is a big restriction on the method (it would imply that our operator T is induced by a point transformation). Let us have a look at what happens when T is a Lamperti operator and S is the ergodic maximal operator $\mathcal{M}f(x)$ associated to T. It is known [29] that T is of the form $Tf(x) = g(x)Uf(x)$, where U is an isomorphism of the sigma-algebra. For simplicity, we will assume that U is just a point transformation. Iterating T we have $T^j f(x) = g_j(x)f(U^j x)$. It is easy to see that $g_{i+j}(x) = g_i(x)g(U^j x)$.

Let $O_\lambda^N = \{x : \mathcal{M}_N f(x) > \lambda\}$. Then $T^j(\chi_{O_\lambda^N})(x) = g_j(x)\chi_{O_\lambda^N}(U^j x)$. This means that $T^j(\chi_{O_\lambda^N})(x) = 0$ unless $U^j(x) \in O_\lambda^N$, in which case it takes the value $g_j(x)$. In other words, $T^j(\chi_{O_\lambda^N}) = g_j\chi_{\{x:\mathcal{M}_N f(U^j x)>\lambda\}}$, which is the same as

$$g_j\chi_{\{x:\mathcal{M}_N f(U^j x)g_j(x)>g_j(x)\lambda\}}\ .$$

If we keep in mind that $T^i f(U^j x)g_j(x) = T^{i+j}f(x)$, we get that

$$T^j(\chi_{O_\lambda^N})(x) = g_j(x)\chi_{\{j:m_N^+ f_x(j)>g_j(x)\lambda\}}(j)\ ,$$

where

$$m_N^+ f(j) = \sup_{0\le n<N}\frac{1}{n+1}\left|\sum_{i=0}^{n} f(j+i)\right|.$$

It is natural then to assume that there exist functions $v_j(x)$ so that for any f and j,

$$T^j\chi_{O_\lambda^N}(x) = v_j(x)\chi_{\{x:R_N T^j f(x)>\lambda v_j(x)\}}(x).$$

We can then write

$$\frac{1}{L+1}\sum_{j=0}^{L+1}\int_X T^j\chi_{O_\lambda^N}h_j(x)=\int_X\frac{1}{L+1}\sum_{\{0\le j\le L:R_Nf_x(j)>\lambda v_j(x)\}}v_j(x)h_j(x)\ .$$

If the operator R_N satisfies the mixed weak type inequality

$$\sum_{\{0\le j\le L:R_Ng(j)>\lambda v_j(x)\}}v_j(x)h_j(x)\le\frac{C}{\lambda}\sum_{j=0}^{L+N}g(j)h_j(x)\ ,$$

then

$$\mu(O_\lambda^N)\le C\frac{1}{\lambda(L+1)}\int_X\sum_{j=0}^{L+N}T^j|f(x)|h_j(x)\,d\mu(x)=C\frac{(L+N+1)}{\lambda(L+1)}\int_X|f|\,d\mu(x).$$

Letting first L, and then N, go to infinity and keeping in mind that the argument works also for $p>1$, we have proved the following.

Theorem 7.3 *Let $1\le p<\infty$. Let T be an operator defined on $L^p(X)$. Let S be an operator induced in some way by T and let us assume that there are operators S_N such that $Sf(x)$ is the monotone limit of the $S_Nf(x)$ and such that:*

(a) T^j commutes with S_N for every j and N.
(b) There exist functions $h_j(x)$ such that

$$\int_X|T^jf(x)|^ph_j(x)d\mu(x)=\int|f(x)|^pd\mu(x).$$

(c) There are operators R_N acting on ℓ_p such that

$$S_N(T^jf)(x)\le CR_Nf_x(j),$$

where $f_x(j)=T^jf(x)$.
(d) If O_λ^N is the set of points where $S_Nf(x)>\lambda$, then there exist functions $v_j(x)$ such that $T^j(\chi_{O_\lambda^N})(x)=v_j(x)\chi_{\{x:R_NT^jf(x)>\lambda v_j(x)\}}(x)$.
(e) The operator R_N satisfies the mixed weak type inequality.

$$\sum_{\{0\le j\le L:R_Ng(j)>\lambda v_j(x)\}}v_j(x)h_j(x)\le\frac{C}{\lambda^p}\sum_{j=0}^{L+N}g(j)h_j(x).$$

Then the operator S is of weak type (p,p).

It is clear that with the obvious changes one can get similar results for the continuous case.

7.3 Back to Dunford–Schwartz

The ergodic theorem of Dunford–Schwartz says that if T is a linear operator which is a contraction in L^1 and in L^∞, then the corresponding maximal operator is of weak type $(1,1)$. We have seen that if we have a theorem about weights in $L^p(\mathbb{Z})$, then Theorem 7.1 implies that for positive operators with positive inverse, in order to obtain an ergodic theorem in L^p one does not need a contraction. A mean-bounded operator is enough. We will now see that in the case of Dunford–Schwartz operators, we could relax the conditions on the operator T assuming only mean boundedness in L^1 and L^∞ if some weighted mixed weak type inequality holds.

Theorem 7.4 *Let T be a linear positive operator with positive inverse. Assume that T is mean bounded in L^∞. Then the functions $j \to g_j(x)$ satisfy A_1^- with a constant independent of x.*

Proof. Taking $f \equiv 1$ we get

$$\frac{g_0(x) + g_1(x) + \cdots + g_n(x)}{n} \le C = C g_0(x) \ .$$

Now if $i \in \mathbb{Z}$,

$$\frac{g_i(x) + \cdots + g_{i+n}(x)}{n} = \frac{g_i(x)(g_0(U^i x) + \cdots + g_n(U^i x))}{n} \le C g_i(x),$$

which is A_1^-.

Theorem 7.5 *Let T be as above. Assume that T is mean bounded in L^1, and let $h(j,x) = T^{*-j}(1)$. Then the functions $j \to h(j,x)$ satisfy A_1^+ with a constant independent of x.*

Proof. From the definition of $h(j,x)$ we get that

$$\int_X (T^j f)(x) h(j,x)\, d\mu(x) = \int_X f(x)\, d\mu(x) \ .$$

From the mean bounded condition we have

$$\big(1 + T^*1 + (T^*)^2 1 + \cdots + (T^*)^n 1\big) \le nC \ .$$

Using now that $(T^*)^{-1}$ is positive, we get

$$\big((T^*)^{-i}1 + (T^*)^{-i+1}1 + (T^*)^{-i+2}1 + \cdots (T^*)^{-i+n}1\big) \le Cn(T^*)^{-i}1 \ ,$$

which may be rewritten as

$$h(i,x) + h(i-1,x) + \cdots + h(i-n,x) \le Cnh(i,x),$$

and this is nothing but A_1^+.

If we had a mixed type inequality of the form

$$\sum_{\{j:m^+f(j)>v(j)\}} uv \le C \sum_{j=-\infty}^{\infty} |f(j)|u(j)$$

for all $f \in l^1(u)$, under the assumptions $u \in A_1^+$ and $v \in A_1^-$, we would obtain that for Lamperti operators the conclusion of the Dunford–Schwartz theorem holds assuming mean boundedness in L^∞ and in L^1. The inequality holds assuming u and v are in A_1 (see [50] in the continuous setting). It follows from this result that if T is a Lamperti operator such that the two-sided averages are mean bounded in L^1 and L^∞, then the maximal operator is of weak type $(1,1)$. Unfortunately, we do not know how to prove the mixed inequality in the one-sided case.

8 The one-sided Hardy–Littlewood maximal operator in dimensions greater than 1

A possible and natural definition of the one-sided Hardy–Littlewood maximal operator in $\mathbb{R}^n$, $n>1$, is

$$M^+f(x) = \sup_{h>0} \frac{1}{h^n} \int_{Q(x,h)} |f|,$$

where $x=(x_1,\dots,x_n)$ and $Q(x,h)=[x_1,x_1+h)\times\cdots\times[x_n,x_n+h)$. Reversing the orientations of each coordinate, we realize that we have 2^n one-sided maximal operators. By symmetry, it suffices to work only with M^+ defined as above. Once we have fixed the operator, we try to characterize the weighted weak type inequality

$$\int_{\{x\in\mathbb{R}^n: M^+f(x)>\lambda\}} w \le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f|^p w \tag{11}$$

by means of a one-sided Muckenhoupt-type condition. The problem is much harder than in dimension 1, and there has not been much progress on this problem for $n>1$ except for the dyadic case. As far as we know, the general problem has not been completely solved. An early discussion can be found in [33], and the characterization of the weak type for a one-sided dyadic maximal operator in $\mathbb{R}^n$ appears in [44]. The characterization in the case $n=2$ has recently been obtained in [21]. We dedicate this section to these results and to showing some of the difficulties that one needs to overcome in order to tackle this problem.

In some sense, the increasing functions play in the one-sided theory the role of the constant functions in the two-sided theory. So, before starting with the general weight theory for M^+, it seems that we would have to know what the behavior of M^+ is for weights which are increasing on each variable. This is not an easy question, and it is not widely known that the increasing weights are good weights for the weighted weak type $(1,1)$ inequality of M^+ in $\mathbb{R}^n$ to hold. We state it as a theorem.

Theorem 8.1 *If w is a nonnegative function on $\mathbb{R}^n$ which is increasing (nondecreasing) on each variable separately, then there exists $C > 0$ such that*

$$\int_{\{x:M^+f(x)>\lambda\}} w \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f|w$$

for all $\lambda > 0$ and all measurable functions f.

This theorem is a direct consequence of the multiparameter Dunford–Schwartz ergodic maximal theorem [18], dated in the 1950s. Observe that if w is a nonnegative function which is increasing on each variable separately, then the semigroup of operators $T^t f(x) = f(x+t)$, $t \in \mathbb{R}^n$, $t = (t_1, \ldots, t_n)$, $t_i > 0$, is a contraction in $L^1(w)$ and in $L^\infty(w)$. These are the key assumptions in the Dunford–Schwartz ergodic theorem. Therefore, we can apply it and the theorem follows. We point out that we do not have any geometric proof of Theorem 8.1.

Recently [21], the problem has been solved only in the case $n = 2$. In that paper, the weights w such that the operator M^+ in $\mathbb{R}^2$ satisfies (11) are characterized by the natural one-sided Muckenhoupt condition. To state the theorem, we need to introduce some notation. If $Q = [a_1 - h, a_1] \times \cdots \times [a_n - h, a_n]$ is a cube with sides parallel to the axis we set $Q^+ = [a_1, a_1 + h] \times \cdots \times [a_n, a_n + h]$. Now, we define the one-sided Muckenhoupt conditions in $\mathbb{R}^n$.

Definition 1 *Let w be a nonnegative measurable functions on $\mathbb{R}^n$. Let $1 < p < \infty$ and let p' be its conjugate exponent, that is, $p + p' = 1$. It is said that w satisfies $A_p^+(\mathbb{R}^n)$, or $w \in A_p^+(\mathbb{R}^n)$, if there exists a positive constant C such that for all cubes Q*

$$\left(\frac{1}{|Q|}\int_Q w\right)^{1/p} \left(\frac{1}{|Q^+|}\int_{Q^+} w^{1-p'}\right)^{1/p'} \leq C.$$

It is said that w satisfies $A_1^+(\mathbb{R}^n)$ if there exists a positive constant C such that for all $h > 0$

$$\frac{1}{h^n}\int_{[x_1-h,x_1]\times\cdots\times[x_n-h,x_n]} w \leq Cw(x) \quad \textit{for almost every } x = (x_1, \ldots, x_n).$$

It is easy to see that if w belongs to the classic Muckenhoupt class $A_p(\mathbb{R}^n)$ and g is a nonnegative function on $\mathbb{R}^n$ which is increasing on each variable separately, then $gw \in A_p^+(\mathbb{R}^n)$. In particular, $g \in A_1^+(\mathbb{R}^n)$.

Now we are ready to state the main theorem in [21].

Theorem 8.2 *Let w be a nonnegative measurable function on $\mathbb{R}^2$. Let $1 \le p < \infty$. Then, the weighted weak type inequality* (11) *holds if and only if $w \in A_p^+\left(\mathbb{R}^2\right)$.*

The proof is geometric and it is based on a covering lemma. The search of this lemma has been inspired by the covering arguments in [44]. It is not clear if this lemma can be extended to higher dimensions. Therefore, the result is restricted to $n = 2$.

Which other strategies can we use to solve the problem? In the one-dimensional case the characterizations of the weighted weak type inequalities for M^+ have been proved using different approaches. The first proof [51] uses the characterization for the Hardy averaging operator and the rising sun lemma; the proofs in [33] use the rising sun lemma and Whitney-type decomposition of an interval; other proofs are based on the fact that the one-sided conditions imply that the one-sided Hardy–Littlewood maximal operator is essentially controlled by the one-sided Hardy–Littlewood maximal operator M_μ^+ with respect to a certain Borel measure.

We have not succeeded in extending these ideas to $\mathbb{R}^n$, $n > 1$. On one hand, we notice that the characterizations for Hardy operators are not well known in dimensions greater than 1; as far as we know, we only have the result in [49], which is restricted to dimension 2. On the other hand, there is not a rising sun lemma in $\mathbb{R}^n$ suitable to work only with cubes, and the Whitney-type decomposition used in $\mathbb{R}$ does not seem to work in $\mathbb{R}^n$.

Finally, if we try to use the last approach we need to know the behavior of the maximal operator M_μ^+ defined as

$$M_\mu^+ f(x) = \sup_{h>0} \frac{1}{\mu([x_1, x_1+h) \times \cdots \times [x_n, x_n+h))} \int_{[x_1,x_1+h)\times\cdots\times[x_n,x_n+h)} |f| d\mu,$$

where $x = (x_1, \ldots, x_n)$ and μ is a Borel measure finite on compact sets (the quotient is understood as 0 if $\mu([x_1, x_1+h_1) \times \cdots \times [x_n, x_n+h_n)) = 0$). We point out that the characterizations of the weighted inequalities for M_μ^+ in dimension 1 were obtained in [2].

We know that the usual maximal operator associated to μ

$$M_\mu f(x) = \sup_{x\in Q} \frac{1}{\mu(Q)} \int_Q |f| d\mu$$

is not always of weak type (1,1) with respect to the measure μ (see [57]) (the supremum is taken over all the cubes such that $x \in Q$). However, the centered maximal operator

$$M_\mu^c f(x) = \sup_{x=\text{center of } Q} \frac{1}{\mu(Q)} \int_Q |f| d\mu$$

is always of weak type (1,1) (now the supremum is taken over all the cubes such that x is the center of Q). We know also [57] that if μ satisfies the doubling condition

$$\mu(2Q) \leq C\mu(Q) \qquad \text{for all cubes } Q,$$

where $2Q$ is the cube with the same center as Q and $|2Q| = 4|Q|$, then M_μ is of weak type (1,1) with respect to the measure μ. Our operator M_μ^+ has a behavior similar to M_μ (see [33]) at least for $n = 2$: If there exists a constant $C > 0$ such that

$$\mu(Q) \leq C\mu(Q^+) \tag{12}$$

for all squares Q, then M_μ^+ is of weak type $(1,1)$ with respect to the measure μ (notice that if $w \in A_p^+(\mathbb{R}^n)$ then the measure $d\mu = w(x)\,dx$ satisfies (12)). The following example shows that M_μ^+ is not always of weak type (1,1): Consider points $z_k = (x_k, y_k)$, $k \geq 1$, such that $y_k = \frac{1}{x_k}$ and $x_k \uparrow 0$. Let $z_0 = (0,0)$ and consider $\mu = \sum_{k=0}^{\infty} \delta_k$, where δ_k is the Dirac measure concentrated at the point z_k. Let B be a ball with center the origin and radius ε (ε small). If $f = \chi_B$ then

$$M_\mu^+ f(z_k) \geq \frac{1}{2}, \quad \text{for all} \quad k \geq 1.$$

Therefore, $\mu(\{M_\mu^+ f \geq \frac{1}{2}\}) = \infty$ and, consequently, we do not have a weak type $(1,1)$ inequality.

Unfortunately, we do not know how to control M^+ by M_μ^+ when $\mu = w(x)\,dx$ and $w \in A_p^+(\mathbb{R}^n)$. Therefore, we have not been able to obtain weighted inequalities for M^+ using the maximal operator M_μ^+.

Acknowledgements

We want to thank Sheldy Ombrosi for enlightening us about Carlos Segovia's work and for giving us the references of his latest papers on one-sided weights.

This research has been partially supported by the Spanish government (Ministerio de Ciencia y Tecnología grants: MTM05-8350-C03-02 and MTM2008-06621-C02-02) and Junta de Andalucía grants (FQM-354 and FQM-01509).

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Lectures on Gas Flow in Porous Media

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Summary. The idea behind the lectures in this chapter is to present, in a relatively simple setting, that of solutions to porous media in one space dimension, several of the main ideas and the main techniques that are at the center of the regularity theory for nonlinear evolution equations and phase transitions. These include exploiting the invariances of the equation to obtain infinitesimal relations and geometric control of the solutions, the role of particular solutions to guide us in our theory and provide us with barriers and asymptotic profiles, the idea of viscosity solutions to a free boundary problem to deduce the geometric properties of the free boundary, and the methods of blowing up solutions and classifying the global profiles to obtain the differentiability properties of a free boundary.

Key words: Porous media equation, free boundary problem, uniform gradient bound, regularity for free boundary.

To the memory of Carlos Segovia, a mentor and a friend.

1 Introduction

The traditional way of modelling phenomena in continuum mechanics is through the description of conservation laws (of mass, energy, etc.) and constitutive relations among the different unknowns, due to the properties of the media or material at hand.

Conservation laws are often introduced as additive set functions, and it is a consequence of the fact that their validity in a very small set implies by superposition their validity in the large, that conservation laws end up as infinitesimal relations on one hand while their being originally set functions implies in turn their divergence structure. The model we are going to consider is described in terms of the gas density $\rho(x,t)$, the velocity field $v(x,t)$, and a pressure $p(x,t)$. The first relation that we will discuss is the conservation of mass: it says that, as time evolves, the amount of mass of the flowing gas in a

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*,
Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_7,

domain G changes proportionally to the mass of the gas flowing through the boundary of G.

Let us consider some given volume G, then the mass (amount) of the gas occupying G at time t is

$$\int_G \rho(x,t)dx.$$

Through the elementary area dS on the boundary of G, the amount of the gas that crosses it per unit of time is $\rho(v \cdot n)dS$, where n is the outward unit normal of ∂G. $v \cdot n$ is positive if the gas flows out of G and negative when it flows into G. The total mass of the gas crossing through ∂G per unit of time is

$$\int_{\partial G} \rho(v \cdot n)dS.$$

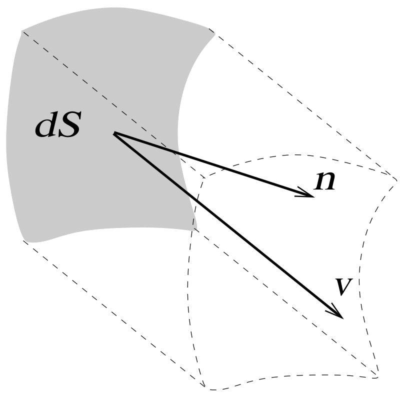

Fig. 1.

On the other hand, the rate of change of the gas in volume G per unit of time is equal to

$$\frac{\partial}{\partial t}\int_G \rho(x,t)dx.$$

Therefore, we may write the conservation of mass as

$$-\frac{\partial}{\partial t}\int_G \rho = \int_{\partial G} \rho v n dS. \tag{1}$$

Hence, after applying the divergence theorem to the right-hand side of this identity and in view of the fact that G is arbitrary, we get $\rho_t + \mathrm{div}\rho v = 0$. This is the equation of conservation of mass.

The next equation comes from a constitutive relation for flow in porous media, known as Darcy's law (named after H. Darcy), stating that v is the gradient of a potential function (the pressure) $v = -Dp$.

Finally, we introduce the *equations of state* $p = \rho^{m-1}, m > 1$ and we get the porous medium equation

$$\rho_t = \operatorname{div}(\rho D \rho^{m-1}),$$

or explicitly,

$$\rho_t = m\rho^{m-1}\Delta\rho + m(m-1)\rho^{m-2}|D\rho|^2. \tag{2}$$

This is a parabolic quasi-linear divergence-type equation. One can define the weak solution of the initial value problem

$$\begin{cases} \rho_t = \Delta(\rho^m) & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \\ \rho(x,0) = \rho_0(x) & x \in \mathbb{R} \end{cases} \tag{3}$$

in a standard manner: ρ is said to be a weak solution of (3) if $D(\rho^m)$ is a distribution and for any $T > 0$ and any smooth $\phi(x,t), \operatorname{supp}\phi(x,t) \subset B_R \times [0,T]$ one has

$$\iint_{\mathbb{R}^n \times [0,T]} [\rho(x,t)\phi_t(x,t) - D\phi(x,t)D(\rho^m(x,t))]dxdt + \int_{\mathbb{R}^n} \rho_0(x)\phi(x,0)dx = 0,$$

where B_R is the ball centered at the origin with radius R, for some $R > 0$.

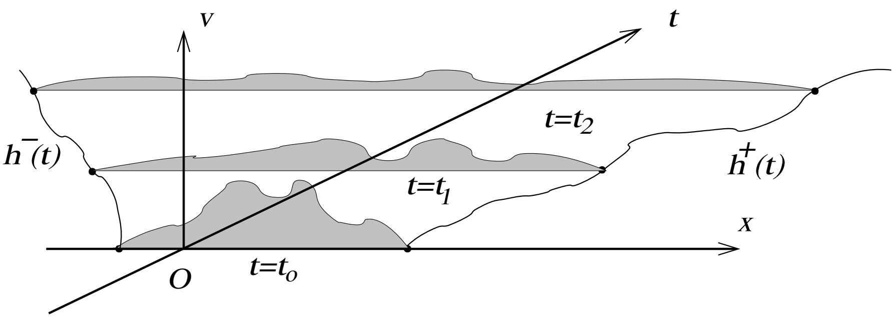

Fig. 2.

Theorem 1.1 *There exists a unique weak solution to the Cauchy problem provided that* $D(\rho_0^m)$ *is bounded. Moreover, a comparison principle holds: if* $\rho_{01}(x) \le \rho_{02}(x)$*, then* $\rho_1(x,t) \le \rho_2(x,t)$*. If the initial data has a compact support, then* $\rho(x,t)$ *has a compact support for every time* t.

The proof of the existence and uniqueness of the weak solutions can be found in [O] (see Publications of Carlos Segovia) and [OKC].

It is helpful in understanding many features of the problem to write the equation satisfied by the pressure p. One of the main reasons is that the particles at the edge of the support of the region occupied by the gas are material points, i.e., they always remain on the moving front, and therefore the speed of the interphase separating gas from vacuum is equal to the speed of the flow Dp. If we consider the normalized pressure

$$v = \frac{m}{m-1}\rho^{m-1}, \tag{4}$$

then v verifies

$$v_t = (m-1)v\Delta v + |Dv|^2. \tag{5}$$

This can be seen logarithmically since $p_t/p = (m-1)\rho_t/\rho$ and

$$\frac{Dp}{p} = (m-1)\frac{D\rho}{\rho},$$

and from the equation

$$\rho_t = \operatorname{div}(\rho Dp) = \rho\Delta p + D\rho Dp,$$

dividing by ρ we obtain

$$\frac{p_t}{p} = \frac{1}{m-1}\Delta p + \frac{|Dp|^2}{p}$$

or

$$p_t = |Dp|^2 + \frac{1}{m-1}p\Delta p.$$

Notice that along the interphase the speed of the material point $x(t)$ is $|Dp| = |\frac{\partial p}{\partial n}|$. Therefore, the speed of the interphase, being the same as that of the material point, becomes

$$\frac{p_t}{p_n} = |\frac{\partial p}{\partial n}|$$

or $p_t = |\frac{\partial p}{\partial n}|^2$, a Hamilton–Jacobi type relation. Formally this means that the term $p\Delta p$ should go to zero at the interphase.

Using these computations and changing p with $\frac{m-1}{m}v$, we obtain

$$v_t = (m-1)v\Delta v + |Dv|^2. \tag{6}$$

In what follows we refer to (5) as the porous medium equation [A], [C2].

To try to understand an evolution problem, one of the first things we should explore are the invariances of the equation and particular solutions. We start by exploring classes of particular solutions. We use the pressure equations. There are three standard types of solutions that we may try.

- Travelling profiles, i.e., solutions that depend only on the variable $x_1 - \alpha t$, α a constant,
- Separation of variables,
- If we have conservation of mass, we can put a Dirac δ (a mass) at the origin and let it go.

1.1 Travelling fronts

Let α be a constant and $(\cdot)_+ = \max(\cdot, 0)$. Then

$$v_\alpha = (\alpha^2 t + \alpha x)_+ \tag{7}$$

is a solution to (5) in the whole space. The free boundary is the line $x = h(t) \equiv -\alpha t$. Note that on the free boundary $x = h(t)$ Darcy's law is satisfied

$$h'(t) = -Dv_\alpha. \tag{8}$$

In the N-dimensional case one can consider

$$v_\alpha = (\alpha^2 t + \alpha(e \cdot x))_+, \ |e| = 1 \tag{9}$$

as a generalization of $(\alpha^2 t + \alpha x)_+$.

1.2 Quadratic solution (separation of variables)

If we try for the solutions of the form $f(x)g(t)$, we find that another explicit solution of (5) in $\mathbb{R}^N \times \mathbb{R}$ is

$$v_p = \frac{1}{2(m+1)} \frac{(x_1^+)^2}{t_0 - t}. \tag{10}$$

This example shows that the free boundary may stay stagnant for quadratic initial data, as shown in Fig. 3 (see Section 5.2).

1.3 Fundamental solution

Let ρ be the Dirac delta at time zero. We expect such a solution to be radially symmetric and self-similar due to the homogeneity of the equation. That reduces the equation to an ordinary differential equations (ODE). More precisely, we must have $\rho(x,t) = W\rho(\frac{x}{M}, 1)$ for some W, M depending on t. But ρ preserves the mass, so

$$\int \rho(x,t)dx = \int \rho(x,1)dx = W \int \rho(\frac{x}{M}, 1)dx = WM^N \int \rho(y,1)dy,$$

hence $W = M^{-N}$. Next, we want ρ to be self-similar, that is, for some constants γ, δ, M,

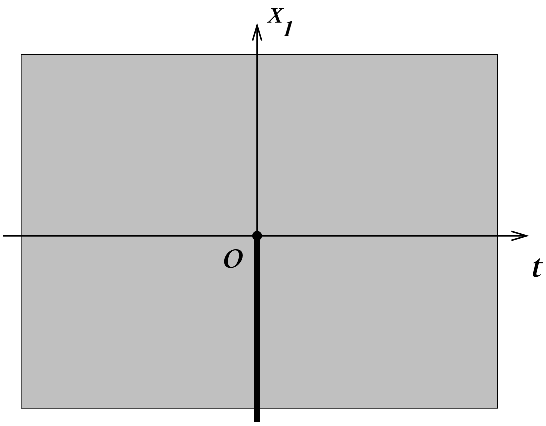

Fig. 3.

$$\rho(x,t) = M^\delta \rho(Mx, M^\gamma t) = \frac{M^\gamma t}{M^2} M^\delta \rho(\frac{x}{M}, 1),$$

implying that M is a power of t, so we seek a solution in the following form:

$$\rho(x,t) = \frac{1}{t^\alpha} F(\frac{x}{t^\beta}).$$

Recall that $p = \rho^{m-1}$. Since ρ is self-similar, then after plugging in $\rho(x,t) = \frac{1}{t^\beta} F(\frac{x}{t^\beta})$ into $\rho_t = \mathrm{div}\rho\nabla p$, all the powers of t will cancel each other, giving

$$\mathrm{div}(F(z)\nabla F^{m-1}(z)) = -N\beta F(z) - \beta\nabla F(z)\cdot z = -\beta \mathrm{div}(zF(z)).$$

Since F is a common factor, it is enough to make

$$\frac{d}{dz}(F^{m-1} - \frac{\beta}{2} z^2) = const.$$

This gives the following solution:

$$v_\delta = \frac{m}{m-1}\frac{1}{t^{\alpha(m-1)}}(a - b\frac{|x|^2}{t^{2\beta}})_+, \tag{11}$$

$$\alpha = N\beta,\ \beta = \frac{1}{2+N(m-1)},\ b = \beta\frac{m-1}{2m}, \tag{12}$$

a is an arbitrary constant. If ρ_δ is the density corresponding to v_δ, then

$$\rho_\delta = \frac{1}{t^\alpha}(a - b\frac{|x|^2}{t^{2\beta}})_+^{\frac{1}{m-1}}.$$

This is the "*fundamental solution*" for the porous medium equation. Note that ρ_δ converges to the Gaussian kernel $t^{-N/2}e^{-\frac{|x|^2}{4t}}$, the fundamental solution of the heat equation, when $m \to 1$ and $a = 1$. Indeed, it is easy to check that the mass of the gas is

$$\begin{aligned} \text{mass} &= \int_{\mathbb{R}^N} \frac{1}{t^\alpha}(a - b\frac{|x|^2}{t^{2\beta}})_+^{\frac{1}{m-1}}\,dx \qquad (13)\\ &= \frac{1}{t^\alpha}\int_{S^N}\int_0^{\sqrt{a}}[a-R^2]_+^{\frac{1}{m-1}}\left[\frac{t^\beta R}{\sqrt{b}}\right]^{N-1}\frac{t^\beta}{\sqrt{b}}dR \\ &= \omega_N b^{-\frac{N}{2}}\int_0^{\sqrt{a}}[a-R^2]_+^{\frac{1}{m-1}}R^{N-1}dR \\ &= \frac{\omega_N}{2}b^{-N/2}B(\frac{m}{m-1},\frac{N}{2}), \end{aligned}$$

where ω_N is the area of the unit sphere and $B(\cdot,\cdot)$ is Euler's beta function. Then using

$$B(\frac{m}{m-1},\frac{N}{2}) \sim \Gamma(\frac{N}{2})(\frac{m}{m-1})^{-\frac{N}{2}}$$

in conjunction with $\omega_N = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}$, we conclude that

$$\begin{aligned} \text{mass} &= \frac{\omega_N}{2}b^{-N/2}B(\frac{m}{m-1},\frac{N}{2}) \\ &\sim \frac{\omega_N}{2}b^{-N/2}\Gamma(\frac{N}{2})(\frac{m}{m-1})^{-\frac{N}{2}} \\ &\sim (\pi\frac{2}{\beta})^{\frac{N}{2}}. \end{aligned}$$

As $\beta \to 1/2$ when $m \to 1$, we get mass $= (2\sqrt{\pi})^N = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}}dx$. For $m = 1$ the heat equation takes the form $\rho_t = \text{div}\rho\nabla\log\rho$.

2 Scaling

All three particular solutions: travelling front, quadratic, and Barenblatt, are self-similar, that is they are invariant under a family of scalings. Let v be a pressure solution to the porous medium equation; then for any A, B, positive constants,

$$v_{A,B} = \frac{B}{A^2}v(Ax, Bt) \qquad (1)$$

is also a solution. If $A = B$ then we call the scaling *hyperbolic*, for $B = A^2$ we call it *parabolic*. Note that the porous medium equation has in some sense as reach a family of scalings than the heat equation. Although nonlinear, it still has two free parameters. The comparison principle with the semigroup generated by the invariant scalings is very useful for obtaining global a priori estimates for dilations of the solution. For instance, we have the following.

Lemma 2.1 *If v is the solution to (5), then*

$$v_t \geq -\frac{v}{t}. \tag{2}$$

Proof. We will compare $v(x,t)$ with $(1+\epsilon)v(x,(1+\epsilon)t)$. Indeed, let $v_{01}(x) = v(x,0), v_{02}(x) = (1+\epsilon)v(x,0)$ for some positive constant ϵ. Then if v_i is the solution to

$$\begin{cases} v_{i,t}(x,t) = (m-1)v_i \Delta v_i + |Dv_i|^2, i = 1,2 \\ v_i(x,0) = v_{0i}(x) \end{cases}, \tag{3}$$

the comparison principle implies

$$v_1 \leq v_2.$$

But $v_2(x,t) = (1+\epsilon)v_1(x,(1+\epsilon)t)$, since we can take $A = 1, B = 1+\epsilon$ as the scaling constants so that

$$v(x,t) \leq (1+\epsilon)v(x,(1+\epsilon)t) = v(x,(1+\epsilon)t) + \epsilon v(x,(1+\epsilon)t), \tag{4}$$

hence

$$\frac{v(x,(1+\epsilon)t) - v(x,t)}{\epsilon} + v(x,(1+\epsilon)t) \geq 0. \tag{5}$$

Letting $\epsilon \to 0$ the result follows. □

This type of argument can be used in many cases for radial symmetry (using infinitesimal rotations) or for monotonicity of solutions (see later the reflection method in Section 5.5)

An important corollary of this lemma is the expansion of the support.

Corollary 2.2 *For $t > t_0$ we have*

$$\frac{v(x_0,t)}{v(x_0,t_0)} \geq e^{ct/t_0}. \tag{6}$$

Hence, if for some point (x_0,t_0) v is positive, then it remains so for any instant of time $t > t_0$.

Proof. By integrating (2) the result follows. □

A much more delicate and beautiful estimate is due to Benilan.

Lemma 2.3 *If v is a solution to the porous medium equation, then one has Benilan's estimate*

$$\Delta v \geq -\frac{1}{(m-1)t}. \tag{7}$$

Note that this estimate implies the previous one, except for the constant.

Proof. Let us assume for a moment that v is smooth. Then applying the Laplacian to both sides of equation (5), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta v) &= (m-1)\Delta(v\Delta v) + \Delta(|Dv|^2) \qquad (8)\\ &= (m-1)v\Delta(\Delta v) + 2mDv\Delta Dv + (m-1)(\Delta v)^2 + 2\sum_{ik} v_{ik}^2.\end{aligned}$$

Set $w = \Delta v$, then w satisfies the partial differential inequality

$$\mathcal{L}(w) = 2\sum_{ik} v_{ik}^2 \geq 0, \qquad (9)$$

where

$$\mathcal{L}(w) = \frac{\partial w}{\partial t} - [(m-1)v\Delta w + 2mDvDw + (m-1)w^2].$$

Due to the presence of v in the equation, the only obvious barrier one can build should be a function of t only, so we want to compare w to a function $-c/t$ for some constant c such that $\mathcal{L}(-c/t) = 0$. In fact, this requires that $c = 1/(m-1)$, hence

$$\mathcal{L}(w) \geq 0 = \mathcal{L}(-\frac{1}{t(m-1)}),$$

while on the boundary we have

$$\Delta v \geq -\infty.$$

Using the comparison principle, the result follows. In the general case, one can approximate (5) by a family of uniformly elliptic equations and then pass to the limit. □

Remark The constant $-1/(m-1)$ is not optimal. Indeed, if we estimate the trace of Hessian $D^2 v$ more carefully, then

$$\frac{(TrD^2v)^2}{N} \leq \sum_{i=1}^{N} v_{ii}^2 \leq \sum_{ik} v_{ik}^2.$$

Thus, introducing

$$\mathcal{L}_1(w) = \frac{\partial w}{\partial t} - [(m-1)v\Delta w + 2mDvDw + (m-1-\frac{2}{N})w^2]$$

and comparing w with $-\frac{1}{t(m-1+\frac{2}{N})}$, we get the sharp form of Benilan's estimate

$$\Delta v \geq -\frac{1}{t(m-1+\frac{2}{N})} = -\frac{\beta N}{t}. \qquad (10)$$

One can check that for the Barenblatt solution this inequality becomes an equality.

Then the immediate consequence of this is the following.

Corollary 2.4 *In the one-dimensional case,*

$$v_x + \frac{x}{(m-1)t} \tag{11}$$

is nondecreasing, so v_x has one-sided limits everywhere. Furthermore, v is semiconvex, so it is locally Lipschitz.

Theorem 2.5 *Let v be a solution to (5). If v is Lipschitz in space, then v is also Lipschitz in time.*

The idea of the proof is very general and can be applied to a more general class of equations. It is again a combination of the scaling invariance of solutions of (5) and maximum principle. First we illustrate the underlying idea for the solutions of the heat equation. Let u be a solution of $\Delta u - u_t = 0$ in a cylinder $Q_\lambda(x_0, t_0)$ and assume that the modulus of continuity of u with respect to x is σ, i.e., $\mathrm{osc}_{x\in B_\lambda(x_0)} u(x,t) \le \sigma(\lambda)$ independently of t. Then the function

$$u_\lambda(x,t) = \frac{u(x_0 + \lambda x, t_0 + \lambda^2 t)}{\sigma(\lambda)}$$

solves the heat equation in the unit cylinder $Q_1(0,0) = B_1 \times (0,1)$ for any $\lambda > 0$. Let us show that then $u_\lambda(0,1) - u_\lambda(0,0) \le c_1$, c_1 depending on σ. If $\epsilon > 0$ and C is a large constant, then $h(x,t) = |x|^2 + 2NCt + 1 + u_\lambda(0,0)$ is a supersolution to the heat equation in the unit cylinder Q_1. Since $\mathrm{osc}_{B_1} u_\lambda \le 1$, we conclude that $u_\lambda(x,0) < h(x,0)$. Assume that the first contact of u_λ and h occurs at the point (x_1, t_1). By the maximum principle, $(x_1, t_1) \in \partial B_1 \times (0,1)$. But then one has

$$1 \ge u_\lambda(x_1,t_1) - u_\lambda(0,0) \ge h(x_1,t_1) - h(0,0) = 1 + 2CNt_1. \tag{12}$$

Hence, u_λ never catches up with h and $u_\lambda < h$ in Q_1 (see Fig. 4). Scaling back, we get that $u(x_0, t_0 + \lambda^2) - u(x_0, t_0) \le c_1\sigma(\lambda)$, and thus setting $\delta = \lambda^2$ we have

$$u(x_0, t_0 + \delta) - u(x_0, t_0) \le c_1 \sigma(\sqrt{\delta}).$$

Using the function $-h$ as a subsolution, we can also prove the lower estimate. In particular, if u is Lipschitz continuous in space, then u is $1/2$ Hölder continuous in time.

A similar argument applies to the solutions of (5), though with a hyperbolic scaling. First we need the following lemma.

Lemma 2.6 *If $v(x_0, t_0) = \alpha$, then $v(x_0, t_0 + h) \le C_1\alpha$ for any $h \le \frac{\alpha}{M}$, where M is a large positive number and C_1 is a positive universal constant.*

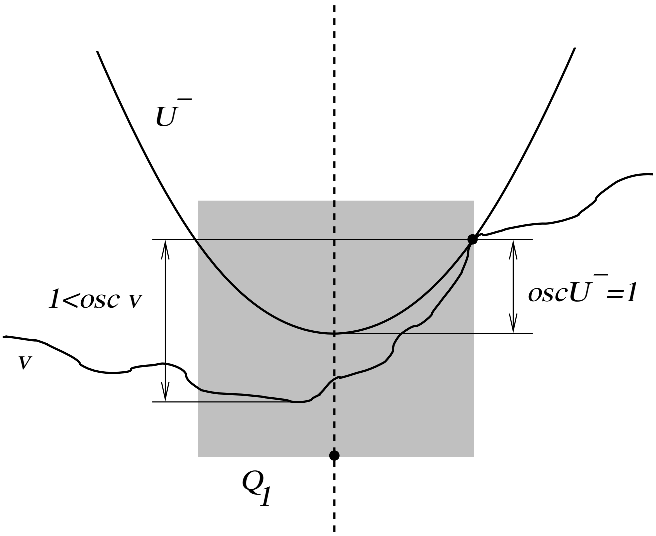

Fig. 4.

Proof. To fix the ideas, let's assume that $(x_0, t_0) = (0, 0)$. Introduce

$$S^- = c\frac{|x|^2 + 2\alpha^2}{\frac{2\alpha}{M} - t}. \tag{13}$$

By a direct computation one can see that S^- is a supersolution to (5) in $\{|x| \leq \alpha\} \times (0, 1)$:

$$S^-_t \geq (m-1)S^- \Delta S^- + |DS^-|^2,$$

provided $c > 0$ is large enough. Indeed, by a direct computation one can see that it is enough to prove $(1 - 4c)|x|^2 < 2\alpha^2(2Nc(m-1) - 1)$ for $|x| \leq \alpha$. Hence, it suffices to assume that $c > 1/(2 + N(m-1))$.

Then

$$S^-(x, 0) = c\frac{|x|^2 + 2\alpha^2}{\frac{2\alpha}{M}} \geq c\alpha M.$$

Since v is Lipschitz in space, we conclude that in $|x| \leq \alpha$

$$\begin{aligned} v(x, 0) &\leq v(0, 0) + L|x| = (1 + L)\alpha \leq c\alpha M \\ &\leq S^-(x, 0), \end{aligned} \tag{14}$$

provided $M > (1 + L)/c$ (this is the relation between M, c, and L-Lipschitz constant). Let t_1 be the first time when S^- touches v at (x_1, t_1). This cannot happen in the interior of cylinder $\{|x| \leq \alpha\} \times [0, \alpha/M]$. From the strong maximum principle, we conclude that $|x_1| = \alpha$, and

$$v(x_1, t_1) = S^-(x_1, t_1) = c\frac{3\alpha^2}{\frac{2\alpha}{M} - t_1}. \tag{15}$$

Furthermore,

$$\begin{aligned} v(x_1, t_1) - v(0, t_1) &\geq S^-(x_1, t_1) - S^-(0, t_1) \\ &= c\frac{3\alpha^2}{\frac{2\alpha}{M} - t_1} - c\frac{\alpha^2}{\frac{2\alpha}{M} - t_1} \\ &\geq c\alpha M. \end{aligned} \tag{16}$$

This contradicts the Lipschitz regularity in space if M is large. Hence, $v < S^- \leq C\alpha$. Note that, using hyperbolic scaling, one can assume that $\alpha = 1$. □

In the same way, using the Barenblatt solution S^+ as a subsolution, one can obtain $v \geq S^+$.

Lemma 2.7 *If $v(x_0, t_0) = \alpha$, then $v(x_0, t_0 + h) - \alpha \geq -C_2\alpha$ for any $h \leq \frac{\alpha}{M}$, where M is a large positive number and C_2 is a positive universal constant.*

Combining these two lemmas, the theorem follows. Next using the scaling and the Lipschitz regularity, we also can prove that Schauder estimates hold in the positivity set.

Theorem 2.8 *Let v be a solution to (5). If $v \sim \alpha$ in $B_\alpha(x_0) \times (t_0, t_0 + \frac{\alpha}{M})$, then*

$$|D^k v| \leq \frac{C(k)}{\alpha^{|k|+1}}, \textit{ in } B_{\alpha/2}(x_0) \times (t_0, t_0 + \frac{\alpha}{2M}).$$

Proof. After scaling, $v_\alpha = v(x_0 + \alpha x, t_0 + \alpha t)/\alpha \sim 1$ in $B_1 \times (0, \frac{1}{M})$ and the equation becomes uniformly parabolic. Then using parabolic Schauder estimates for v_α and scaling back, the result follows. □

3 Regularity of the free boundary

We will now illustrate the main steps in proving the free boundary regularity for our problem. That is: the (increasing) boundary of the support of v may stay stationary for a while, but as soon as it starts to move, it will always have positive speed. In fact, its speed will satisfy a differential inequality and it will be a C^1 curve. The two main ingredients that reappear in much more complex problems are already present here: an asymptotic convexity of the free boundary under dilations and the possibility to classify global profiles. The two main barriers we will use are the pressure form of the fundamental solution and the travelling fronts.

First we observe the following property of the Barenblatt solutions. Let

$$v_1(x) = \frac{m}{m-1}(A - B|x|^2)_+.$$

Recall that the Barenblatt solution in N dimensions is

$$v(x,t) = \frac{m}{m-1}\frac{1}{t^{\alpha(m-1)}}(A - B\frac{|x|^2}{t^{2\beta}})_+ \tag{1}$$

$$\alpha = N\beta, \qquad \beta = \frac{1}{2+(m-1)N}, \qquad B = \frac{(m-1)\beta}{2m}$$

and $A > 0$ is the constant which determines the total mass. Hence, $v(x,t)$ is the solution to

$$\begin{cases} v_t = (m-1)v\Delta v + |Dv|^2, & t > 1 \\ v(x,1) = v_1(x). \end{cases} \tag{2}$$

A direct computation ($N = 1$) then shows that, on the free boundary $x = h(t)$,

$$\begin{aligned} h'(1) &= \beta\sqrt{\frac{A}{B}} \\ h''(1) &= -(1-\beta)h'(1). \end{aligned} \tag{3}$$

We now consider a solution $v(x,t)$, with initial data $v_0 = (x, t_0)$ supported in the interval $[a,b]$, then the free boundary for $t \geq t_0 > 0$ consists of two monotone, Lipschitz curves $h^+(t), h^-(t)$. More precisely, we have the following.

Lemma 3.1 *Let* $h(t) \equiv h^+(t) = \sup\{x, v(x,t) > 0\}$. *Then* $h(t)$ *is monotone and Lipschitz.*

Proof. $h(t)$ is monotone since by (6) we have

$$v(\bar{x},t) \geq v(\bar{x},\bar{t})e^{-ct/\bar{t}} > 0,$$

provided $v(\bar{x},\bar{t}) > 0$. To prove that $h(t)$ is Lipschitz, we compare v with a travelling front solution. Recall that v is Lipschitz for $N = 1$. Let x_0, t_0 be a free boundary point. From the mean value theorem,

$$v(x,t_0) = -v_x(\cdot,t_0)(h(t_0) - x) \leq C(h(t_0) - x), \qquad x < x_0 = h(t_0).$$

Now consider the wave solution

$$v_\alpha = \alpha(\alpha(t - t_0) + (x_0 - x))_+.$$

Then if $\alpha = C$, the Lipschitz constant, and applying the comparison principle, we conclude that $v(x,t) \leq v_\alpha(x,t), t > t_0, x > x_0$. Hence, the free boundary of v is inside the free boundary of v_α, so the slope of h is controlled by the Lipschitz constant of v. □

Remark v can be controlled from above by a travelling front. Next we shall see that v can be controlled from below by a Barenblatt solution. In turn this will imply a formula for a speed $h'(t)$ of the free boundary.

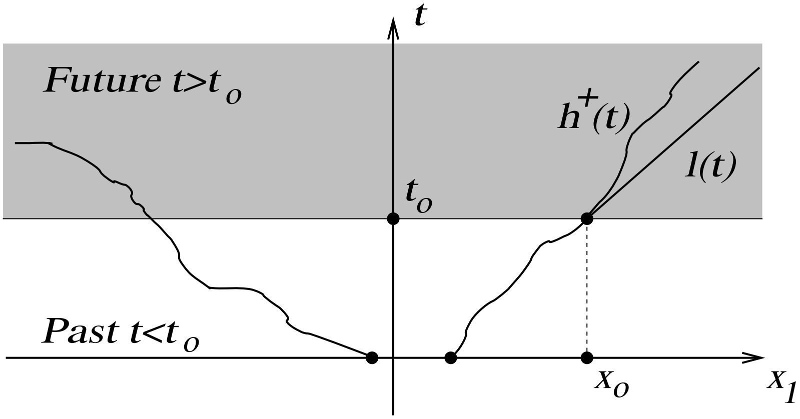

Fig. 5.

Corollary 3.2 *If $v(x_0, t_0) > 0$ and $v_x(x_0, t_0) = -\alpha$, then there is a parabola $P(x)$ such that $v \geq P(x)$, $t \geq t_0$ and $P'' = \beta/2, P'(x_0) = -\alpha, P(x_0) = v(x_0, t_0)$.*

Proof. If it is necessary, we may consider the scaled function $\bar{v}(x,t) = \frac{1}{t_0} v(xt_0, tt_0)$ and we may assume that $t_0 = 1$. Then by (10) of Section 2,

$$\Delta \bar{v} = \bar{v}_{xx} \geq -\beta,$$

and therefore

$$\bar{v}(x,t) + \beta \frac{|x - \bar{x}_0|^2}{2}$$

is convex. Then if $\ell(x)$ is the support plane at the point $\bar{x}_0 = t_0 x_0$, then $\bar{v} \geq P(x)$, where

$$\begin{aligned} P(x) &= -\beta \frac{|x - \bar{x}_0|^2}{2} + \ell(x - \bar{x}_0) \\ &= -\beta \frac{|x - \bar{x}_0|^2}{2} - \alpha(x - \bar{x}_0) + \bar{v}(\bar{x}_0, \bar{t}_0) \\ &= \frac{m}{m-1}(-b|x - \bar{x}_0|^2 - 2Nb(x - \bar{x}_0)) + \bar{v}(\bar{x}_0, \bar{t}_0) \\ &= \frac{m}{m-1}(bN^2 - b|x - \bar{x}_0 + N|^2) + \bar{v}(\bar{x}_0, \bar{t}_0). \end{aligned} \tag{4}$$

This is the Barenblatt solution truncated at $t = 1$. Note that the free boundary condition (3) $h' = \alpha$ is satisfied. Scaling back to the original variables, the result follows. □

Corollary 3.3 *Let (x_0, t_0) be a free boundary point. For $t \geq t_0$, v is above the corresponding Barenblatt solution.*

Proof. Without loss of generality, we can assume that $t_0 = 1$. From the previous corollary, we have the $P(x)$ is the Barenblatt solution truncated at $t = 1$. Since all parabolas are below v, have the same second derivatives, and v_x is semicontinuous, then the conclusion of the corollary holds for free boundary points as well (see Fig. 6). Indeed, we can approach the free boundary point (x_0, t_0) a little bit from the future or a little bit from the past. Since everything is continuous, we can pass to the limit and get a desired limit parabola $P(x)$, which is the truncated Barenblatt, touching v from below at the free boundary point. Then let v_P be the Barenblatt corresponding to the initial condition $P(x)$. Thus, by the comparison principle, $v_P \le v, t > 1$. □

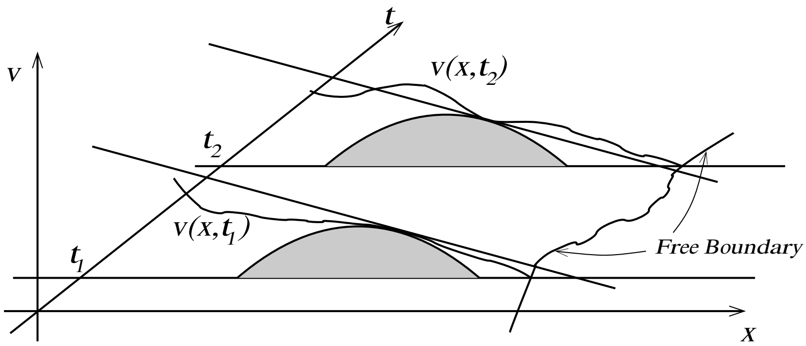

Fig. 6.

The next lemma makes precise the free boundary condition, which heuristically is Darcy's law. Notice that the lim sup below is taken for both (x, t) converging to (x_0, t_0), both from the past and the future.

Lemma 3.4 *Let (x_0, t_0) be a free boundary point and let*

$$\alpha = \limsup_{(x,t)\to(x_0,t_0)} (-v_x). \tag{5}$$

Then for $t \ge t_0$ we have

$$h(t) = x_0 + \alpha(t - t_0) + \omega(t - t_0). \tag{6}$$

Further, from above we only have that $\omega = o(t - t_0)$, but from below we have the stronger inequality

$$\omega(t - t_0) \ge -\alpha C(t - t_0)^2 + o((t - t_0)^2).$$

Proof. From the previous result, it follows that

$$h(t) \ge h(t_0) + \alpha(t - t_0) - C\alpha(t - t_0)^2.$$

To show that the reversed inequality is satisfied, we take $\epsilon > 0$ and use the definition of α,

$$-v_x \leq (\alpha + \epsilon), \qquad x_0 - \delta < x < x_0.$$

Then from the Lipschitz continuity and the mean value theorem, we get

$$v(x, t_0) = -v_x(\cdot)(x_0 - x) \leq -(\alpha + \epsilon)(x - x_0)$$

in the future.

Lemma 3.5 *Coming from the past* $h(t) \geq x_0 + \alpha(t - t_0) - o(t - t_0)$.

Proof. Assume that for a sequence $t_k \uparrow t_0$

$$x_k = h(t_k) \leq x_0 + (\alpha + \delta)(t_k - t_0).$$

Since

$$\limsup(-v_x) = \alpha,$$

we have that $(-v_x) \leq \alpha + \delta/4$ in a small enough neighborhood of (x_0, t_0), $N_s(x_0, t_0)$, in space and time. We will compare v with the travelling front solution $w(x, t)$, going through (x_k, t_k) with speed $\alpha + \delta/2$. From the estimate by above of x_k, this wave goes through the left of x_0 at t_0 and thus crosses the free boundary. But if we go backwards in x from x_k at time t_k, we have that

$$w(x, t_k) \geq v(x, t_k) + \frac{\delta}{4}|x - x_k|.$$

So for $x = x_k - s$, $w(x_k - s, t_k) \geq v(x_k - s, t_k) + \frac{\delta}{4}h$. This is enough room to go into the future starting at $x_k - s, t_k$ to use w as a barrier for v in the region $\{x \geq x_k - s, t_k \leq t \leq t_0\}$ and get a contradiction. □

We will now get a differential inequality for h. We start by proving that $h(t)$ is a "viscosity subsolution" of $h'' \geq Ch'$.

Lemma 3.6 *If* $h(t)$ *has at* $t = t_0$ *a tangent parabola* $x(t) = \ell(t) + a(t - t_0)^2$, *by above then* a *must be* $a \geq -C\ell'$.

Proof. At t_0, h has a tangent line with a slope α, from Lemmas 3.5 and 3.4. Therefore, ℓ' must be equal to α. If $a \leq -C\ell' = -C\alpha$ going into the future, we have a contradiction to Lemma 3.4. □

We are now in the final step. In this section we want to illustrate how to show the regularity of a free boundary by classifying global "blow-outs" of a solution. We already have an important fact. We know that the free boundary of the blow-out must be convex. We will now show that every blow-out is a travelling front solution and go back and deduce that the free boundary was indeed C^1.

Consider the travelling front $v_\alpha = (\alpha + \epsilon)[(\alpha + \epsilon)(t - t_0) - (x - x_0)]_+$. Then from the comparison principle, $v \leq v_\alpha$, $t > t_0$, therefore

$$h(t) \leq h(t_0) + (\alpha + \epsilon)(t - t_0) + o((t - t_0)^2).$$

At this point, at least coming from the future we seem to have the differential inequality

$$h'' \geq -Ch'$$

that heuristically would imply that h is "quasi convex," i.e., $h(t)+Ct^2$ should be convex in the neighborhood of t_0. In this opportunity, we introduce a new idea, the idea of "viscosity solution," i.e., using comparisons with smooth super and subsolutions.

In one dimension the idea is straightforward, as we will see below. In more dimensions it has become very fruitful to show that very weak solutions of an equation are actually smooth.

Corollary 3.7 *There exists a large positive constant C depending on Lipschitz norm of v such that*

$$\phi(t) = h(t) + Ct^2 \tag{7}$$

is convex.

Proof. If not, find a parabola touching h with $a \leq -Ch''$.

Corollary 3.8 *$h(t)$ satisfies to*

$$h''(t) \geq -Ch'(t), \tag{8}$$

in the viscosity sense. Hence,

$$h'(t) \geq h'(t_0)e^{-c(t-t_0)}.$$

4 Differentiability of the free boundary

We want to show that h is actually differentiable. Since $h(t) + Ct^2$ is convex, it has left and right differentials at every point, and for $t < s$,

$$(h'(t))^- \leq (h'(t))^+ \leq (h'(s))^- \leq (h'(s))^+.$$

To fix the ideas, we assume that the origin is on the free boundary.

4.1 Blow-up

For $\lambda > 0$ consider function

$$v_\lambda(x,t) = \frac{v(\lambda x, \lambda t)}{\lambda}.$$

It follows that v_λ is a solution to the porous medium equation. Moreover, v_λ is Lipschitz; therefore, $\lim_{\lambda \to 0} v_\lambda = v^\infty$ exists and it is called the blow-up of v. Note that

- second derivative

$$(v_\lambda)_{xx} = \lambda v_{xx}(\lambda x, \lambda t) \geq -\frac{\lambda}{(m-1)t} \to 0,$$

so v^∞ is convex.

- free boundary $h(t)$ is convex and consists of two lines

$$h(t) = \begin{cases} At, t > 0 \\ Bt, t < 0 \end{cases} \tag{1}$$

with $A \geq B \geq 0$.

Note that

$$v^\infty(x,0) = \lim \frac{v(x_0 + \lambda x, t_0 + \lambda t)}{\lambda} = 0, x > 0, \tag{2}$$
$$v_x^\infty(x,0) = \lim v_x(x_0 + \lambda x, t_0 + \lambda t) = -A,$$

i.e., $v^\infty(x,0) = (-Ax)_+$. Therefore, we conclude from the uniqueness theorem that $v^\infty(x,t) = A(At - x)_+$ for $t > 0$.

We want to show now that the travelling front cannot "break" going into the past. To do this we will go far to the left for $t = t_0$ and get a contradiction. We start with an estimate for the decay of v_{tt}.

Lemma 4.1 *There exists a constant $C > 0$ such that for $x_0 < 0$ large we have*

$$v_{tt}^\infty(x,t) \leq \frac{C}{|x_0|}, \forall (x,t) \in \{|x - x_0| \leq \frac{|x_0|}{2}, |t| \leq \frac{|x_0|}{M}\} \equiv D. \tag{3}$$

Proof. Let us consider the scaled function

$$v_R(x,t) = \frac{v^\infty(Rx, Rt)}{R}, R = \frac{1}{|x_0|}.$$

Then v_R is Lipschitz in $|t| \leq \frac{1}{M}, |x - x_0| \leq 1/2$. By parabolic Schauder estimates, we have that

$$|(v_R)_{tt}| \leq C, \tag{4}$$

and returning to v^∞ the result follows. □

Corollary 4.2

$$\lim_{x \to -\infty} [v^\infty(x,t) - A(At - x)_+] = 0.$$

Proof. Take $x < 0$ large, then at $(x,0)$ v^∞ is C^1 smooth. Therefore, using Taylor's formula,

$$v^\infty(x,t) - A(At - x) = v^\infty(x,0) + tv^\infty(x,0) + \frac{t^2}{2} v_{tt}^\infty(\cdot) - A(At - x) \tag{5}$$
$$= \frac{t^2}{2} v_{tt}^\infty(\cdot) \to 0,$$

when $x \to -\infty$. □

4.2 Classification of the global solutions

Next, we want to show that $v^\infty = A(At - x)$. An important step to prove this, is to show that at any point

$$v_x \geq -A.$$

Assume that for some (x_0, t_0) we have $-v_x^\infty(x_0, t_0) = -A - \delta < -A$; then we can put under v a travelling front with speed $A + \delta$ that will catch up with the free boundary. If $x < x_0 < 0$ we have

$$\begin{aligned} v^\infty(x,t) &= v^\infty(x_0,t) - v_x^\infty(\cdot)(x_0 - x) \\ &\geq v^\infty(x_0,t) + (A+\delta)(x_0 - x). \end{aligned} \tag{6}$$

We used $v_x^\infty(\cdot) \leq v_x^\infty(x_0, t)$ since $v_{xx}^\infty \geq 0$.

Thus,

$$\begin{aligned} 0 \leftarrow v^\infty(x,t) - A(At - x) &\geq v^\infty(x_0,t) + (A+\delta)(x_0 - x) - At^2 + Ax \\ &= v^\infty(x_0,t) - A^2 t + Ax_0 + \delta(x_0 - x) \to +\infty, \end{aligned} \tag{7}$$

provided $\delta > 0$, and this is a contradiction, hence

$$v_x^\infty \geq -A.$$

This implies that for any (x, t)

$$v^\infty(x,t) \geq A(At - x).$$

Finally, let us show that v^∞ is the wave function $A(At-x)$. Take a point $(\bar{x}, \bar{t})$ and assume that $\bar{x} \leq A\bar{t}$ such that $v^\infty(\bar{x}, \bar{t}) > A(A\bar{t} - \bar{x})$, which contradicts the strong maximum principle. Next, assume that $\bar{x} > A\bar{t}$. But then for $t > \bar{t}$ we know that $v^\infty(\bar{x}, t) > 0$ by (6). Contradiction. □

5 Remarks

5.1 *N*-dimensional results

In the N-dimensional case, v may not be Lipschitz, though it is always Hölder continuous. In [CVW] the authors proved Lipschitz continuity for large times. More precisely, if T_0 is the time when the support of $v(x,t)$ overflows the smallest ball, where the initial support is contained, then v is Lipschitz in $\mathbb{R}^N \times (\tau, \infty)$ for any $\tau > T_0$, with bounds depending on the initial data and τ. Also supp v is bounded for any t but eventually it covers the whole space. As a consequence, the free boundary is Lipschitz. Furthermore, it is also $C^{1,\alpha}$ [CW]. However, there is an example constructed by J. Graveleau showing that if supp v_0 has holes, then Dv may blow up. Therefore, the result in [CVW] is optimal.

5.2 Waiting time

As the example of the quadratic solution indicates, the free boundary may stay stagnant. If there exists a $t^\star \in [0,T]$ so that $h(t)$ does not move for $t \in (0,t^\star)$ and $h(t)$ moves for $t > t^\star$, then $t^\star$ is called the waiting time. Note that when h starts moving it never stops. The value of $t^\star$ depends on the initial condition. The next theorem is due to Knerr [K].

Theorem 5.1 *If initial data* $v_0(x) \geq c(-x)^\gamma, -\delta < x < 0$ *for some* $\gamma \in (0,2)$*, then* $t^\star = 0$*. If* $v_0(x) \leq cx^2, -\delta < x < 0$*, then* $t^\star > 0$*.*

If $t \in (t^\star, T)$ then $h \in C^1(t^\star, T)$ [CF], and hence the free boundary condition is satisfied in the classical sense.

Theorem 5.2 *Let* $t_m = 1/2(m+1)$ *and let* v *be the solution of*

$$\begin{cases} v_t = (m-1)v\Delta v + |Dv|^2 & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ v(x,0) = v_0(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

If for some $\alpha, \beta > 0$

$$\begin{aligned} v_0(x) \leq \alpha x^2 + o(x^2) \ \textit{as}\ x \uparrow 0 \\ v_0(x) = 0, \forall x \in \mathbb{R}^+ \\ v_0(x) \leq \beta x^2 \ \textit{in}\ x \in \mathbb{R}^+, \end{aligned} \tag{2}$$

then

$$\frac{t_m}{\beta} \leq t^\star \leq \frac{t_m}{\alpha},$$

in particular, if $\alpha = \beta$ $t^\star = t_m/\alpha$*.*

5.3 Viscosity solutions

Viscosity solutions were introduced by M. Crandall and P. Lions in the context of the first order equations of Hamilton–Jacobi type.

For instance, if we are given in the interval $[-1,1]$ the equations

$$\begin{cases} |w_x| = 1 \\ w(-1) = w(1) = 0 \end{cases}$$

any zig-zag with slopes 1 and -1 would be a candidate for a weak solution. But there are two natural ones: $w = 1 - |x|$ and $-w = |x| - 1$. The solution w is selected by the "vanishing viscosity" method, i.e., it is the limit of w^ϵ, solutions to

$$\epsilon w^\epsilon_{xx} + (1 - |w^\epsilon_x|) = 0,$$

(thus the name viscosity solution). They realized that w would be "touched by below" by a smooth function ϕ only if $|\phi'| \geq 1$, and by above only if $|\phi'| \leq 1$ (i.e., it is the most concave solution).

It was soon realized that this was an excellent way to define weak solutions for equations in nondivergent form, i.e., defined by a comparison with a "specific profiles" (quadratic polynomials for second order PDEs, global profiles for phase transition problems, etc.)

Here we sketch how the theory works for the Laplacian [CC].

Definition 5.3 *A function $u : \mathbb{R}^N \to \mathbb{R}$, continuous in Ω, is said to be a subsolution (supersolution) to $\Delta u = 0$, and we write $\Delta u \geq 0$ ($\Delta u \leq 0$), if the following holds: if $x_0 \in \Omega, \phi \in C^2(\Omega)$ and $u - \phi$ has local maximum (minimum) at x_0, then $\Delta \phi \geq 0$ ($\Delta \phi \leq 0$). A solution is a function u which is both a subsolution and a supersolution.*

Heuristically this definition says that a subsolution cannot touch a solution from below. Indeed, assume that ϕ is a C^2 *strict* subsolution touching the solution u from below at x_0, then $u(x) \leq \phi(x), u(x_0) = \phi(x_0)$. Since ϕ touches u from below, then $D^2 u \geq D^2 \phi$. On the other hand, $0 < \Delta \phi \leq \Delta u = 0$, a contradiction. To make this argument work for subsolutions, one needs to consider $\phi + \epsilon |x|^2$ and let $\epsilon \downarrow 0$. Another way of checking this is to use the maximum principle, for we have from previous computation that $\Delta \phi = \Delta u = 0$ and $\phi - u$ has a local minimum at x_0; thus by the maximum principle, $u = \phi$.

If in the definition one changes Δu with $F(D^2 u)$, then the definition of viscosity solutions for elliptic operator F follows.

As an example let us show that any continuous of $\Delta u = 0$ is a classical harmonic function. In the one-dimensional case, the classical solution is a line $\ell(x)$. Now if u is above this line, then bringing the parabola $P(x) = \ell(x) - \varepsilon x^2$ from infinity will touch u at some point which will contradict $P_{xx} \geq 0$.

In the N-dimensional case, let B_ρ be a ball of radius ρ, and let $\Delta u = 0$ in B_ρ in the viscosity sense. Let v be harmonic in B_ρ and $u = v$ on ∂B_ρ. Then v is the Poisson integral of the continuous function u. Thus, $v \in C^2(B_\rho)$. We want to compare u with v in B_ρ, choosing ρ to be sufficiently small.

Denote $M = \max_{B_\rho}(u - v)$ and suppose that $M > 0$. Then for some x_0, $M = u(x_0) - v(x_0)$ and x_0 is an interior point, since $u = v$ on the boundary of B_ρ. Consider $u_\varepsilon(x) = v(x) + \varepsilon(x - x_0)^2$, where ε is a small positive number. Let $M_1 = \max_{B_\rho}(u - v + \varepsilon(x - x_0)^2)$ and it is attained at some x_1. Then if we choose ε to be very small, we have that x_1 should be close to x_0, that is, x_1 is an interior point, then we have

$$\begin{aligned} u(x) &\leq v(x) + \varepsilon(x - x_0)^2 + M_1, \\ u(x_1) &= v(x_1) + \varepsilon(x_1 - x_0)^2 + M_1, \end{aligned} \tag{3}$$

hence $\phi(x) = v(x) + \varepsilon(x - x_0)^2 + M_1$, which is C^2 in B_ρ, touches u at x_1 from above. From the definition of the viscosity solutions, $0 \leq \Delta \phi = \Delta(v - \varepsilon(x - x_0)^2) = -2N\varepsilon$, a contradiction. Thus, x_0 is not an interior point and $u \leq v$ in B_ρ.

In the same way one can show that $m = \min_{B_\rho}(u - v) \geq 0$. Otherwise, if $m < 0$ and the minimum is attained at some point y_0, we will compare u to the function $\psi(x) = v - \varepsilon(x-y_0)^2 + m_1$, where $m_1 = \min_{B_rho}(u - v - \varepsilon(x-y_0)^2)$ and the inequality $u \geq v$ follows.

In this section, we give an idea of how the techniques described in the lectures surface in the theory of minimal surfaces and free boundary problems.

5.4 Global profiles and regularity

A minimal surface is a surface which has the smallest area among all surfaces with a prescribed boundary condition. Classical solutions to minimal surface problems do not always exist. Therefore, one has to seek the solution in a weak sense, that is, to define the area in some generalized way. This is given in a weak fashion through the divergence theorem, by means of the *perimeter*. Ω is said to be a set of finite perimeter if for any smooth vector field ψ, $\sup_{x\in\Omega}|\psi| \leq 1$ compactly supported in Ω,

$$|\int_\Omega \mathrm{div} v| \leq C_0.$$

The best constant C_0 is called the perimeter of set $\partial\Omega$. Then the perimeter is semicontinous under L^1 convergence of characteristic functions χ_Ω. Note that heuristically, using the divergence theorem,

$$|\int_\Omega \mathrm{div} v| = |\int_{\partial\Omega} v\cdot\nu| \leq area(\partial\Omega).$$

Sets of finite perimeter can also be thought of as L^1 limits of polyhedra with a uniformly finite area. Then we can look at the following problem:

Among all sets of finite perimeter $\Omega \subset B_1$, find one which has minimum perimeter.

The existence of a set with minimal perimeter is immediate by compactness.

Having defined the generalized area and generalized minimal surface, one tries to explore how "classical" it can be. In other words, we try to show that except for an unavoidable singular set Σ, it is a smooth hypersurface satisfying an equation of mean curvature.

One of the ways of doing so is to exploit the invariance of area minimizing surfaces (such as scaling!) and a monotonicity formula. The latter is the following: if S is an area minimizing surface and $0 \in S$ in $\mathbb{R}^{N+1}$, then

$$A(r) = \frac{area(S\cap B_r)}{r^N}$$

is a monotone function of r. Moreover, if $A(r)$ is identically constant, S has to be a cone, i.e., the defining function is homogeneous. One then considers the sequence of dilations $S_k = \{x, r_k x \in S\}, r_k \downarrow 0$. The "limiting blow-up" object is a surface S_0, also called the global solution, for which $A(r) = const. = A(0^+)$. Hence, if one can classify all possible minimal cones S_0 which are alternatives to a hyperplane, a regularity theorem can be deduced. For instance, if $N < 8$ the only such cones are hyperplanes and the generalized minimal surface S is really an analytic graph.

Many free boundary problems can be treated parallel to the theory of minimal surfaces [CS]. For instance, consider the classical two phase problem [ACF].

Let u be a Lipschitz function in unit ball B_1, such that

$$\begin{gathered} \Delta u = 0, \text{ in } \{u > 0\} \cup \{u < 0\}, \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 \text{ on } \mathcal{F} = \partial\{u > 0\}. \end{gathered} \tag{4}$$

The free boundary here is $\mathcal{F}$ and the extra gradient jump condition $(u_\nu^+)^2 - (u_\nu^-)^2 = 1$ on $\mathcal{F}$ is satisfied in some weak sense. A weak solution of this problem can be obtained by minimizing the functional

$$J(u) = \int_{B_1} |Du|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}}$$

for some positive constants λ_+, λ_-. Note that if $\Lambda = \lambda_+^2 - \lambda_-^2 > 0$, then

$$J(u) = \int_{B_1} |Du|^2 + \Lambda \chi_{\{u>0\}} + \lambda_-^2 |B_1|,$$

so $J(u)$ is the sum of Dirichlet energy and Λ meas$\{u > 0\}$. This suggests that the fact that u is a minimizer imposes some minimality on the volume of the positivity set. It turns out that $\partial\{u > 0\}$ is a generalized surface of positive mean curvature [C1], i.e., if $\partial\{u > 0\}$ is perturbed inside of positivity set $\{u > 0\}$ near B_r, then for perturbed surface S', $H^{n-1}(S') \geq H^{n-1}(\partial\{u > 0\})$.

Let us illustrate how the ideas from minimal surface theory can be applied to classify the global solutions of (4) in the two-dimensional case. Assume that u is a minimizer of J so that it solves (4) in some weak sense. First note that Lipschitz is the best possible regularity for u one can expect in view of the gradient jump along the free boundary $\mathcal{F}$. Using a monotonicity formula, one can show that u is Lipschitz [ACF]. For $r_k \downarrow 0$ and $0 \in \partial\{u > 0\}$ let us consider $u_k(x) = u(r_k x)/r_k$. This function is well defined for Lipschitz function u. Then $S_k = \partial\{u_k > 0\}$ and $S_0 = \partial\{u_0 > 0\}$, where $u_0 = \lim u_k$ must be a homogeneous global solution. If the free boundary $\partial\{u_0 > 0\}$ forms an angle with aperture θ at zero, then $\partial\{u_0 > 0\}$ is a cone Γ_θ with aperture θ. We want to show that there are no alternatives to u_0 being a linear function, i.e., Γ_θ is a half-plane.

By rotation of the coordinate system, we may assume that $\Gamma_\theta = \{x \in \mathbb{R}^2 : 0 < x_2 < x_1 \tan\theta\}$. Let us write the Laplacian in polar coordinates,

$$\Delta u = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_r) + \frac{1}{r} u_{\phi\phi}\right).$$

Since $u = rg(\phi)$, $g(\phi)$ verifies the Cauchy problem

$$\begin{cases} g''(\phi) + g(\phi) = 0 \\ g(0) = g(\theta) = 0, \end{cases} \tag{5}$$

which has a unique solution $g(\phi) = \sin\phi$. Therefore, $\theta = \pi$ and Γ_θ is the upper half-plane. Hence, in the two-dimensional case all the global solutions are linear functions and $\mathcal{F}$ is differentiable.

5.5 Moving plane method

As a final example of the power of symmetries, we describe the moving plane method created by A.D. Aleksandrov in his study of surfaces of constant curvature. The well-known theorem of A.D. Aleksandrov states: if S is a surface of constant nonzero mean curvature, then S is a sphere. Let's take a one-parameter family of planes and move it in some constant direction. Let S_t be the surface which is a reflection of S with respect to the plane corresponding to t. Then at some point, S_t and S would be tangent to each other; hence, by the Hopf lemma, $S = S_t$, so S is symmetric with respect to any plane, and thus it is a sphere.

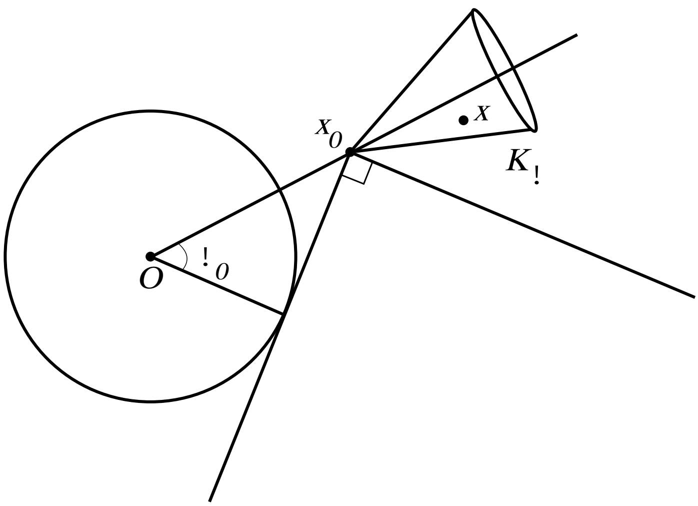

Fig. 7.

This technique can be used to prove Lipschitz continuity of the free boundary in the N-dimensional case when $\operatorname{supp} u_0 \subset B_1$ and the free boundary is strictly outside of B_1. Let $\Omega = \operatorname{supp} u_0(x)$ and let $a = \inf_{x\in\Omega} x_1, b = \sup_{x\in\Omega} x_1$, where $x = (x_1, x'), x' \in \mathbb{R}^{N-1}$. Then for any $\lambda \in (a, b)$ consider $x_\lambda = 2\lambda - x_1$. So x_λ is the reflection of x with respect to the plane $x_1 = \lambda$. Our goal is to show that $u(x,t) \geq u(x_\lambda, t), t > 0$ when $a > 0$ or $b < 0$. Indeed, $u_0(x) \geq u_0(x_\lambda)$ and $u_0(x) = u_0(x_\lambda)$ for $x_1 = \lambda$; hence, the comparison principle applies. In particular, this implies monotonicity of u in the x_1 direction since λ is an arbitrary number in (a, b). Clearly, this reflection argument applies to any plane $\Sigma = \{x \in \mathbb{R}^N, (x - y_0) \cdot \ell = 0\}$ for some fixed point y_0 and unit direction ℓ, provided that Ω has a positive distance from Σ. Now to prove that the boundary of $\operatorname{supp} u$ is Lipschitz, it is enough to show that there exists a uniform cone of monotonicity at each point on the boundary of the support of $u(x, t)$ when the free boundary lies outside B_1.

Now take $x_0, |x_0| > 1$ and let $K_\alpha = \{x, \angle(x - x_0, x_0) \leq \alpha\}, \alpha < \alpha_0$ with $\cos \alpha_0 = 1/|x_0|$. Then for any plane Σ reflating x to x_0 we can apply Aleksandrov's idea and conclude that in K_α u is monotone. See Fig. 7. Notice that we did not assume Lipschitz regularity for u.

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Sharp Global Bounds for the Hessian on Pseudo-Hermitian Manifolds

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Summary. We consider an abstract CR manifold equipped with a strictly positive definite Levi form, which defines a pseudo-Hermitian metric on the manifold. On such a manifold it is possible to define a natural sums of squares sub-Laplacian operator. We use Bochner identities to obtain Cordes–Friedrichs type inequalities on such manifolds where the L^2 norm of the Hessian tensor of a function is controlled by the L^2 norm of the sub-Laplacian of the function with a sharp constant for the inequality. By perturbation we proceed to develop a Cordes–Nirenberg type theory for non-divergence form equations on CR manifolds. Some applications are given to the regularity of p-Laplacians on CR manifolds.

Key words: CR manifolds, Friedrichs inequalities, sub-Laplacian, Bochner identities, p-Laplacian, Alexandrov–Bakelman–Pucci estimate, Cordes–Nirenberg estimates.

Dedicated to the memory of our friend and colleague Carlos Segovia.

1 Introduction

In partial differential equation (PDE) theory, harmonic analysis enters in a fundamental way through the basic estimate valid for $f \in C_0^\infty(\mathbb{R}^n)$, which states,

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^n)} \leq c(n,p) \|\Delta f\|_{L^p(\mathbb{R}^n)}, \text{ for } 1 < p < \infty. \tag{1}$$

This estimate is really a statement of the L^p boundedness of the Riesz transforms, and thus (1) is a consequence of the multiplier theorems of Marcinkiewicz and Hörmander–Mikhlin, [15]. More sophisticated variants of

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*,
Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_8,

(1) can be proved by relying on the square function [15] and [14]. In particular, (1) leads to *a priori* $W^{2,p}$ estimates for solutions of

$$\Delta u = f, \text{ for } f \in L^p. \tag{2}$$

Knowledge of $c(p, n)$ allows one to perform a perturbation of (2) and study

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f, \tag{3}$$

as was done by Cordes [5], where $A = (a^{ij})$ is bounded, measurable, elliptic, and close to the identity in a sense made precise by Cordes. The availability of the estimates of Alexandrov–Bakelman–Pucci and the Krylov–Safonov theory [8] allows one to obtain estimates for (3) in full generality without relying on a perturbation argument. See also [12].

Our focus here will be to study the CR analog of (3). Since at this moment there is no suitable Alexandrov–Bakelman–Pucci estimate for the CR analog of (3), we will be seeking a perturbation approach based on an analog of (1) on a CR manifold. Our main interest is the case $p = 2$ in (1). In this case a simple integration by parts suffices to prove (1) in $\mathbb{R}^n$. We easily see that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^2(\mathbb{R}^n)}^2 = \|\Delta f\|_{L^2(\mathbb{R}^n)}^2 . \tag{4}$$

In the case of (1) on a CR manifold, a result has been recently obtained by Domokos–Manfredi [7] in the Heisenberg group. The proof in [7] makes use of the harmonic analysis techniques in the Heisenberg group developed by Strichartz [16] which will not apply to studying such inequalities for the Hessian on a general CR manifold, although other nilpotent groups of step 2 can be treated similarly [6].

Instead, we shall proceed by integration by parts and use of the Bochner technique. A Bochner identity on a CR manifold was obtained by Greenleaf [9] and will play an important role in our computations.

We now turn to our setup. We consider a smooth orientable manifold M^{2n+1}. Let $\mathcal{V}$ be a vector sub-bundle of the complexified tangent bundle $\mathbb{C}TM$. We say that $\mathcal{V}$ is a CR bundle if

$$\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}, \; [\mathcal{V}, \mathcal{V}] \subset \mathcal{V}, \text{ and } \dim_{\mathbb{C}} \mathcal{V} = n. \tag{5}$$

A manifold equipped with a sub-bundle satisfying (5) will be called a CR manifold. See the book by Trèves [18]. Consider the sub-bundle

$$H = \operatorname{Re}\left(\mathcal{V} \oplus \overline{\mathcal{V}}\right). \tag{6}$$

H is a real $2n$-dimensional vector sub-bundle of the tangent bundle TM. We assume that the real line bundle $H^\perp \subset T^*M$, where T^*M is the cotangent bundle, has a smooth non-vanishing global section. This is a choice of a non-vanishing 1-form θ on M and (M, θ) is said to define a pseudo-Hermitian structure. M is then called a pseudo-Hermitian manifold. Associated to θ we have the Levi form L_θ given by

$$L_\theta(V, \overline{W}) = -i\, d\theta(V \wedge \overline{W}), \text{ for } V, W \in \mathcal{V}. \tag{7}$$

We shall assume that L_θ is definite and orient θ by requiring that L_θ be positive definite. In this case, we say that M is strongly pseudo-convex. We shall always assume that M is strongly pseudo-convex.

On a manifold M that carries a pseudo-Hermitian structure, or a pseudo-Hermitian manifold, there is a unique vector field T, transverse to H defined in (6) with the properties

$$\theta(T) = 1 \quad \text{ and } \quad d\theta(T, \cdot) = 0. \tag{8}$$

T is also called the Reeb vector field. The volume element on M is given by

$$dV = \theta \wedge (d\theta)^n. \tag{9}$$

A complex valued 1-form η is said to be of type $(1, 0)$ if $\eta(\overline{W}) = 0$ for all $W \in \mathcal{V}$, and of type $(0, 1)$ if $\eta(W) = 0$ for all $W \in \mathcal{V}$.

An admissible co-frame on an open subset of M is a collection of $(1, 0)$ forms $\{\theta^1, \ldots, \theta^\alpha, \ldots, \theta^n\}$ that locally form a basis for $\mathcal{V}^*$ and such that $\theta^\alpha(T) = 0$ for $1 \leq \alpha \leq n$. We set $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. We then have that $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ locally form a basis of the complex co-vectors, and the dual basis is the complex vector fields $\{T, Z_\alpha, \overline{Z_\alpha}\}$. For $f \in C^2(M)$ we set

$$Tf = f_0, \qquad Z_\alpha f = f_\alpha, \qquad \overline{Z_\alpha} f = f_{\overline{\alpha}}. \tag{10}$$

We note that in what follows all our functions f will be real valued.

If follows from (5), (7), and (8) that we can express

$$d\theta = i\, h_{\alpha\overline{\beta}}\, \theta^\alpha \wedge \theta^{\overline{\beta}}. \tag{11}$$

The Hermitian matrix $(h_{\alpha\overline{\beta}})$ is called the Levi matrix.

On pseudo-Hermitian manifolds Webster [19] has defined a connection, with connection forms ω_α^β and torsion forms $\tau_\beta = A_{\beta\alpha}\theta^\alpha$, with structure relations

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau_\beta, \qquad \omega_{\alpha\overline{\beta}} + \omega_{\bar{\beta}\alpha} = dh_{\alpha\overline{\beta}} \tag{12}$$

and

$$A_{\alpha\beta} = A_{\beta\alpha}. \tag{13}$$

Webster defines a curvature form

$$\prod\nolimits_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta,$$

where we have used the Einstein summation convention. Furthermore, in [19] it is shown that

$$\prod{}_{\alpha}^{\beta} = R_{\alpha\bar{\beta}\rho\bar{\sigma}}\theta^{\rho}\wedge\theta^{\bar{\sigma}} + \text{ other terms.}$$

Contracting two indices using the Levi matrix $(h_{\alpha\bar{\beta}})$, we get

$$R_{\alpha\bar{\beta}} = h^{\rho\bar{\sigma}}\ R_{\alpha\bar{\beta}\rho\bar{\sigma}}. \tag{14}$$

The Webster–Ricci tensor $\mathrm{Ric}(V,V)$ for $V\in\mathcal{V}$ is then defined as

$$\mathrm{Ric}(V,V) = R_{\alpha\bar{\beta}}x^{\alpha}\overline{x^{\beta}},\ \text{for } V=\sigma_{\alpha}x^{\alpha}Z_{\alpha}. \tag{15}$$

The torsion tensor is defined for $V\in\mathcal{V}$ as follows:

$$\mathrm{Tor}(V,V) = i\left(A_{\bar{\alpha}\bar{\beta}}\overline{x^{\alpha}}\bar{x}_{\beta} - A_{\alpha\beta}x^{\alpha}x^{\beta}\right). \tag{16}$$

In [19], Prop. (2.2), Webster proves that the torsion vanishes if $\mathcal{L}_T$ preserves H, where $\mathcal{L}_T$ is the Lie derivative. In particular, if M is a hypersurface in $\mathbb{C}^{n+1}$ given by the defining function ρ,

$$\mathrm{Im}z_{n+1} = \rho(z,\overline{z}), \qquad z=(z_1,z_2,\dots,z_n), \tag{17}$$

then Webster's hypothesis is fulfilled, and the torsion tensor vanishes on M. Thus, for the standard CR structure on the sphere S^{2n+1} and on the Heisenberg group, the torsion vanishes.

Our main focus will be the sub-Laplacian Δ_b. We define the horizontal gradient ∇_b and Δ_b as follows:

$$\nabla_b f = \sum_{\alpha} f_{\overline{\alpha}}Z_{\alpha}, \tag{18}$$

$$\Delta_b f = \sum_{\alpha} f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}. \tag{19}$$

When $n=1$ we will need to frame our results in terms of the CR Paneitz operator. Define the Kohn Laplacian $\Box_b$ by

$$\Box_b = \Delta_b + iT. \tag{20}$$

Then the CR Paneitz operator P_0 is defined by

$$P_0 f = \left(\overline{\Box}_b\Box_b + \Box_b\overline{\Box}_b\right)f - 2\left(Q+\overline{Q}\right)f, \tag{21}$$

where

$$Qf = 2i\,(A^{11}f_1)_1.$$

See [10] and [4] for further details.

2 The main theorem

Theorem 1 *Let M^{2n+1} be a strictly pseudo-convex pseudo-Hermitian manifold. When M is non-compact, assume that $f \in C_0^\infty(M)$. When M is compact with $\partial M = \emptyset$, we may assume $f \in C^\infty(M)$. When f is real valued and $n \geq 2$, we have*

$$\sum_{\alpha,\beta} \int_M |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \int_M \Big(Ric + \frac{n}{2}\,Tor\Big)(\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{2n} \int_M |\Delta_b f|^2. \quad (1)$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$\int_M |f_{11}|^2 + |f_{1\bar{1}}|^2 + \int_M \left(Ric - \frac{3}{2}\,Tor\right)(\nabla_b f, \nabla_b f) \leq \frac{3}{2} \int_M |\Delta_b f|^2. \quad (2)$$

Here by $\sum_{\alpha,\beta} |f_{\alpha\beta}|^2$ we mean the Hilbert–Schmidt norm square of the tensor $f_{\alpha\beta}$ and similarly for $\sum_{\alpha,\beta} |f_{\alpha\bar{\beta}}|^2$.

Proof We begin by noting the Bochner identity established by Greenleaf, Lemma 3 in [9]:

$$\begin{aligned} \frac{1}{2}\Delta_b\left(|\nabla_b f|^2\right) &= \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \mathrm{Re}\left(\nabla_b f, \nabla_b(\Delta_b f)\right) \\ &\quad + \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b, \nabla_b) + i \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha} 0}\right), \end{aligned} \quad (3)$$

where for $V, W \in \mathcal{V}$ we use the notation $(V, W) = L_\theta(V, \overline{W})$ and $|V| = (V, V)^{1/2}$. Using the fact that $f \in C_0^\infty(M)$ or if $\partial M = \emptyset$, M is compact, integrate (3) over M using the volume ((9) of Section 1) to get

$$\begin{aligned} &\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) \\ &\quad + i \int_M \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha} 0}\right) = - \int_M \mathrm{Re}\left(\nabla_b f, \nabla_b(\Delta_b f)\right). \end{aligned} \quad (4)$$

Integration by parts in the term on the right yields (see (5.4) in [9])

$$- \int_M \mathrm{Re}(\nabla_b f, \nabla_b(\Delta_b f)) = \frac{1}{2} \int_M |\Delta_b f|^2. \quad (5)$$

Combining (4) and (5) we get

$$\begin{aligned} &\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \int_M \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) \\ &\quad + i \int_M \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha} 0}\right) = \frac{1}{2} \int_M |\Delta_b f|^2. \end{aligned} \quad (6)$$

To handle the third integral on the left-hand side, we use Lemmas 4 and 5 of [9] (valid for real functions) according to which we have

$$i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) = \frac{2}{n}\int_M \Big(\sum_{\alpha,\beta} \big(|f_{\alpha\bar{\beta}}|^2 - |f_{\alpha\beta}|^2 \big) - \mathrm{Ric}(\nabla_b f, \nabla_b f) \Big), \quad (7)$$

and

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) = &- \frac{4}{n}\int_M \big| \sum_\alpha f_{\alpha\bar{\alpha}} \big|^2 \\ &+ \frac{1}{n}\int_M |\Delta_b f|^2 \\ &+ \int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \quad (8)$$

Applying the Cauchy–Schwarz inequality to the first term on the right-hand side of (8), we get

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) \geq &- 4\int_M \sum_{\alpha,\beta} |f_{\alpha\bar{\beta}}|^2 \\ &+ \frac{1}{n}\int_M |\Delta_b f|^2 \\ &+ \int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \quad (9)$$

Multiply (7) by $1-c$ and (9) by c , $0 < c < 1$, and where c will eventually be chosen to be $1/(n+1)$, and add to get

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) \geq &\ 2\ \frac{(1-c)}{n}\int_M \sum_{\alpha,\beta} \big(|f_{\alpha\bar{\beta}}|^2 - |f_{\alpha\beta}|^2 \big) \\ &- 2\frac{(1-c)}{n}\int_M \mathrm{Ric}(\nabla_b f, \nabla_b f) \\ &- 4c\int_M \sum_{\alpha,\beta} |f_{\alpha\overline{\beta}}|^2 \\ &+ \frac{c}{n}\int_M |\Delta_b f|^2 + c\int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \quad (10)$$

We now insert (10) into (6) and simplify. We have

$$\begin{aligned}&\left(1-\frac{2(1-c)}{n}\right)\int_M \mathrm{Ric}(\nabla_b f,\nabla_b f)+\\&\left(\frac{(n-2)}{2}+c\right)\int_M \mathrm{Tor}(\nabla_b f,\nabla_b f)+\\&\left(1+\frac{2(1-c)}{n}-4c\right)\int_M\sum_{\alpha,\beta}|f_{\alpha\bar\beta}|^2+\\&\qquad\left(1-\frac{2(1-c)}{n}\right)\int_M\sum_{\alpha,\beta}|f_{\alpha\beta}|^2\le\left(\frac12-\frac{c}{n}\right)\int_M|\Delta_b f|^2.\end{aligned}\tag{11}$$

Let $c = 1/(n+1)$. Then (11) becomes

$$\left(\frac{n-1}{n+1}\right)\left[\int_M\sum_{\alpha,\beta}\left(|f_{\alpha\beta}|^2+|f_{\alpha\bar\beta}|^2\right)+\int_M\left(\mathrm{Ric}+\frac{n}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)\right]\le\left(\frac{n-1}{n+1}\right)\left(\frac{n+2}{2n}\right)\int_M|\Delta_b f|^2.\tag{12}$$

Since $n \geq 2$, $n-1>0$ and we can cancel the factor $\frac{n-1}{n+1}$ from both sides to get (1).

We now establish (2) using some results by Li–Luk [11] and [4]. When $n = 1$, identity (6) becomes

$$\int_M|f_{1\bar1}|^2+|f_{11}|^2+\int_M\left(\mathrm{Ric}-\frac12\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)+i\int_M(f_{10}f_{\bar1}-f_{\bar10}f_1)=\frac12\int_M|\Delta_b f|^2.\tag{13}$$

By (3.8) in [11] we have

$$i\int_M(f_{01}f_{\bar1}-f_{0\bar1}f_1)=-\int_M f_0^2.$$

Moreover, by (3.6) in [11] we also have

$$i\,(f_{10}f_{\bar1}-f_{\bar10}f_1)=i\,(f_{01}f_{\bar1}-f_{0\bar1}f_1)+\mathrm{Tor}(\nabla_b f,\nabla_b f),$$

and combining the last two identities we get

$$i\int_M(f_{10}f_{\bar1}-f_{\bar10}f_1)=-\int_M f_0^2+\int_M\mathrm{Tor}(\nabla_b f,\nabla_b f).\tag{14}$$

Substituting (14) into (13) we obtain

$$\int_M|f_{1\bar1}|^2+|f_{11}|^2+\int_M\left(\mathrm{Ric}+\frac12\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)-\int_M f_0^2=\frac12\int_M|\Delta_b f|^2.\tag{15}$$

Next, we use (3.4) in [4],

$$\int_M f_0^2 = \int_M |\Delta_b f|^2 + 2\int_M \mathrm{Tor}(\nabla_b f, \nabla_b f) - \frac{1}{2}\int_M P_0 f\cdot f. \tag{16}$$

Finally, substitute (16) into (15) and simplify to get

$$\int_M |f_{1\bar{1}}|^2 + |f_{11}|^2 + \int_M \left(\mathrm{Ric} - \frac{3}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) + \frac{1}{2}\int_M P_0 f\cdot f = \frac{3}{2}\int_M |\Delta_b f|^2.$$

Assuming $P_0 \geq 0$ we obtain (2). □

We now wish to make some remarks about our theorem:

(a) It is shown in [7] that on the Heisenberg group the constant $(n+2)/2n$ is sharp. Since the Heisenberg group is a pseudo-Hermitian manifold with Ric $\equiv 0$ and Tor $\equiv 0$, we easily conclude that our theorem is sharp and contains the result proved in [7].

(b) We notice that when we consider manifolds such that $\mathrm{Ric}+(n/2)\mathrm{Tor} > 0$, then for $n \geq 2$, in general we have the strict inequality

$$\sum_{\alpha,\beta}\int_M |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 < \frac{n+2}{2n}\int_M |\Delta_b f|^2.$$

On the Heisenberg group Ric $\equiv 0$, Tor $\equiv 0$, and the constant $(n+2)/2n$ is achieved by a function with fast decay [7]. Thus, the Heisenberg group is, in a sense, extremal for inequality (1) in Theorem 1. A similar remark holds for inequality (2).

(c) The hypothesis on the Paneitz operator in the case $n = 1$ in our theorem is satisfied on manifolds with zero torsion. A result from [2] shows that, if the torsion vanishes, the Paneitz operator is non-negative.

(d) We note that Chiu [4] shows how to perturb the standard pseudo-Hermitian structure in $\mathbb{S}^3$ to get a structure with non-zero torsion, for which $P_0 > 0$ and $\mathrm{Ric} - (3/2)\mathrm{Tor} > 1$. To get such a structure, let θ be the contact form associated to the standard structure on $\mathbb{S}^3$. Fix g a smooth function on $\mathbb{S}^3$. For $\epsilon > 0$ consider

$$\tilde{\theta} = e^{2f}\theta, \text{ where } f = \epsilon^3 \sin(\frac{g}{\epsilon}). \tag{17}$$

Since the sign of the Paneitz operator is a CR invariant and θ has zero torsion, we conclude by [2] that the CR Paneitz operator $\tilde{P}_0$ associated to $\tilde{\theta}$ satisfies $\tilde{P}_0 > 0$. Furthermore, following the computation in Lemma (4.7) of [4], we easily have for small ϵ that

$$\mathrm{Ric} - \frac{3}{2}\mathrm{Tor} \geq (2 + O(\epsilon))\, e^{-2f} \geq 1 \geq 0.$$

Thus, the hypothesis of the case $n = 1$ in our theorem are met, and for such $(M, \tilde{\theta})$ we have, for $f \in C^\infty(M)$, the estimate

$$\int_M |f_{11}|^2 + |f_{1\bar{1}}|^2 \, dV \leq \frac{3}{2} \int_M |\Delta_b f|^2 \, dV.$$

(e) Compact pseudo-Hermitian 3-manifolds with negative Webster curvature may be constructed by considering the co-sphere bundle of a compact Riemann surface of genus g, $g \geq 2$. Such a construction is given in [3].

3 Applications to PDE

For applications to subelliptic PDE it is helpful to restate our main result Theorem 1 in its real version. We set

$$X_i = \mathrm{Re}(Z_i) \text{ and } X_{i+n} = \mathrm{Im}(Z_i)$$

for $i = 1, 2, \ldots, n$. The horizontal gradient of a function is the vector field

$$\mathfrak{X}(f) = \sum_{i=1}^{2n} X_i(f) X_i.$$

Its sub-Laplacian is given by

$$\Delta_{\mathfrak{X}} f = \sum_{i=1}^{2n} X_i X_i(f),$$

and the horizontal second derivatives are the $2n \times 2n$ matrix

$$\mathfrak{X}^2 f = (X_i X_j(f)) \, .$$

For f real we have the following relationships:

$$\nabla_b f = \mathfrak{X}(f) + i \left(\sum_{i=1}^{n} X_i(f) X_{i+n} - X_{i+n}(f) X_i \right),$$

$$\Delta_b f = 2 \, \Delta_{\mathfrak{X}} f,$$

and

$$\sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 = 2 \sum_{i,j} |X_i X_j(f)|^2 = 2 |\mathfrak{X}^2 f|^2,$$

where the expression on the right is the Hilbert–Schmidt norm of the tensor taken by viewing the Levi form as a Riemannian metric on H.

Theorem 2 *Let M^{2n+1} be a strictly pseudo-convex pseudo-Hermitian manifold. When M is non-compact, assume that $f \in C_0^\infty(M)$. When M is compact with $\partial M = \emptyset$, we may assume $f \in C^\infty(M)$. When f is real valued and $n \geq 2$, we have*

$$\int_M |\mathfrak{X}^2 f|^2 + \int_M \frac{1}{2}\left(Ric + \frac{n}{2}\,Tor\right)(\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{n}\int_M |\Delta_{\mathfrak{X}} f|^2. \quad (1)$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$\int_M |\mathfrak{X}^2 f|^2 + \int_M \frac{1}{2}\left(Ric - \frac{3}{2}\,Tor\right)(\nabla_b f, \nabla_b f) \leq 3\int_M |\Delta_{\mathfrak{X}} f|^2. \quad (2)$$

Let $A(x) = (a_{ij}(x))$ a $2n \times 2n$ matrix. Consider the second order linear operator in non-divergence form

$$\mathcal{A}u(x) = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u(x), \quad (3)$$

where coefficients $a_{ij}(x)$ are bounded measurable functions in a domain $\Omega \subset M^{2n+1}$. Cordes [5] and Talenti [17] identified the optimal condition expressing how far $\mathcal{A}$ can be from the identity and still be able to understand (3) as a perturbation of the case $A(x) = I_{2n}$, when the operator is just the sub-Laplacian. This is called Cordes condition, roughly says that all eigenvalues of A must cluster around a single value.

Definition 1 *([5],[17], [7]) We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that*

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2n} a_{ij}^2(x) \leq \frac{1}{2n-1+\varepsilon}\left(\sum_{i=1}^{2n} a_{ii}(x)\right)^2 \quad (4)$$

for a.e. $x \in \Omega$.

Let $c_n = \frac{(n+2)}{n}$ for $n \geq 2$ and $c_1 = 3$ be the constants in the right-hand sides of the equations of Theorem 2. We can now adapt the proof of Theorem 2.1 in [7] to get

Theorem 3 *Let M^{2n+1} be a strictly pseudo-convex pseudo-Hermitian manifold such that $Ric + \frac{n}{2}\,Tor \geq 0$ if $n \geq 2$ and $Ric - \frac{3}{2}\,Tor \geq 0$, $P_0 \geq 0$ if $n = 1$. Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{(1-\varepsilon)c_n} < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in C_0^\infty(\Omega)$ we have the a priori estimate*

$$\|\mathfrak{X}^2 u\|_{L^2} \leq \sqrt{1+\frac{2}{n}}\,\frac{1}{1-\gamma}\,\|\alpha\|_{L^\infty}\|\mathcal{A}u\|_{L^2}\,, \quad (5)$$

where

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} = \frac{\sum_{i=1}^{2n} a_{ii}(x)}{\sum_{i,j=1}^{2n} a_{ij}^2(x)}\,.$$

Proof We start from formula (2.7) in [7] which gives

$$\int_{\Omega} |\Delta_{\mathfrak{X}} u(x) - \alpha(x)\mathcal{A}u(x)|^2 \, dx \leq (1-\varepsilon) \int_{\Omega} |\mathfrak{X}u|^2 dx.$$

We now apply Theorem 2 to get

$$\int_{\Omega} |\Delta_{\mathfrak{X}} u(x) - \alpha(x)\mathcal{A}u(x)|^2 \, dx \leq (1-\varepsilon)c_n \int_{\Omega} |\Delta_{\mathfrak{X}} f|^2.$$

The theorem then follows as in [7]. □

Remark. The hypothesis of Theorem 2, $n \geq 2$, can be weakened to assume only a bound from below,

$$\mathrm{Ric} + \frac{n}{2}\mathrm{Tor} \geq -K, \text{ with } K > 0,$$

to obtain estimates of the type

$$\int_M |\mathfrak{X}^2 f|^2 \leq \frac{(n+2)}{n} \int_M |\Delta_{\mathfrak{X}} f|^2 + 2K \int_M |\mathfrak{X} f|^2. \tag{6}$$

A similar remark applies to the case $n = 1$.

We finish this paper by indicating how the *a priori* estimate of Theorem 3 can be used to prove regularity for p-harmonic functions in the Heisenberg group $\mathcal{H}^n$ when p is close to 2. We follow [7], where full details can be found. Recall that, for $1 < p < \infty$, a p-harmonic function u in a domain $\Omega \subset \mathcal{H}^n$ is a function in the horizontal Sobolev space

$$W^{1,p}_{\mathfrak{X},\mathrm{loc}}(\Omega) = \{u \colon \Omega \mapsto \mathbb{R} \text{ such that } u, \mathfrak{X}u \in L^p_{\mathrm{loc}}(\Omega)\}$$

such that

$$\sum_{i=1}^{2n} X_i \left(|\mathfrak{X}u|^{p-2} X_i u \right) = 0\,, \text{ in } \Omega \tag{7}$$

in the weak sense. That is, for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} |\mathfrak{X}u(x)|^{p-2} (\mathfrak{X}u(x), \mathfrak{X}\phi(x) \, dx = 0. \tag{8}$$

Assume for the moment that u is a smooth solution of (7). We can then differentiate to obtain

$$\sum_{i,j=1}^{2n} a_{ij} \, X_i X_j u = 0\,, \text{ in } \Omega, \tag{9}$$

where

$$a_{ij}(x) = \delta_{ij} + (p-2)\frac{X_i u(x)\, X_j u(x)}{|\mathfrak{X}u(x)|^2}\,.$$

A calculation shows that this matrix satisfies the Cordes condition (4) precisely when

$$p-2 \in \left(\frac{n - n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\,,\ \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\right). \tag{10}$$

In the case $n = 1$ this simplifies to

$$p-2 \in \left(\frac{1-\sqrt{5}}{2}\,,\ \frac{1+\sqrt{5}}{2}\right).$$

We then deduce *a priori* estimates for $\mathfrak{X}^2 u$ from Theorem 3. To apply the Cordes machinery to functions that are only in $W^{1,p}_{\mathfrak{X}}$, we need to know that the second derivatives $\mathfrak{X}^2 u$ exist. This is done in the Euclidean case by a standard difference quotient argument applied to a regularized p-Laplacian. In the Heisenberg case this would correspond to proving that solutions to

$$\sum_{i=1}^{2n} X_i\left(\left(\frac{1}{m} + |\mathfrak{X}u|^2\right)^{\frac{p-2}{2}} X_i u\right) = 0 \tag{11}$$

are smooth. Contrary to the Euclidean case (where solutions to the regularized p-Laplacian are C^∞-smooth), in the subelliptic case this is known only for $p \in [2, c(n))$, where $c(n) = 4$ for $n = 1, 2$, and $\lim_{n\to\infty} c(n) = 2$ (see [13]). The final result will combine the limitations given by (10) and $c(n)$.

Theorem 4 *(Theorem 3.1 in [7]) For*

$$2 \le p < 2 + \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}$$

we have that p-harmonic functions in the Heisenberg group $\mathcal{H}^n$ are in $W^{2,2}_{\mathfrak{X},loc}(\Omega)$.

At least in the one-dimensional case $\mathcal{H}^1$ one can also go below $p = 2$. See Theorem 3.2 in [7]. We also note that when p is away from 2, for example, $p > 4$ nothing is known regarding the regularity of solutions to (7) or its regularized version (11) unless we assume *a priori* that the length of the gradient is bounded below and above

$$0 < \frac{1}{M} \le |\mathfrak{X}u| \le M < \infty.$$

See [1] and [13].

Acknowledgements

S.C. was supported in part by NSF Award DMS-0600971. J.J.M. was supported in part by NSF award DMS-0500983. S.C. wishes to thank Shri S. Devananda and Shri Raghavendra for their encouragement in a difficult moment.

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Recent Progress on the Global Well-Posedness of the KPI Equation

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Summary. In this chapter I survey the well-posedness theory for the Kadomtsev–Petviashvili (KPI) equation, culminating with the recent proof (joint with A. Ionescu and D. Tataru) of the global well-posedness of the KPI equation in two space dimensions, with data in the natural energy space.

Key words: KPI equation, energy space, global well-posedness.

Consider the initial value problem for the Kadomtsev–Petviashvili (KP) equations:

$$\begin{cases} \partial_t u + \partial_x^3 u \pm \partial_y^2 \partial_x^{-1} u + \partial_x \left(\frac{u^2}{2}\right) = 0 \\ u|_{t=0} = u_0(x,y), & (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} \end{cases} \tag{KP}$$

(here and in the sequel, ∂_x^{-1} denotes the primitive in the x-variable).

The equation with the $+$ sign is called KPII, while the one with the $-$ sign is called KPI. These equations model propagation along the x-axis of nonlinear dispersive long waves on the surface of a fluid, with a slow variation along the y-axis (see [KP]). Both equations are two-dimensional generalizations of the well-known Korteweg–de Vries equation, which are still completely integrable (see [AC]). In this chapter we will be concerned with their well-posedness theory. It turns out that the standard energy method can be used for both KPI and KPII, to establish local well-posedness in high order Sobolev spaces ([U], [IN]). In light of the progress made for the Korteweg–de Vries equation, using harmonic analysis methods (see, for instance, [K2] for a recent survey), it became an important issue to prove low-regularity local and global well-posedness for (KP). It turns out that the situation for KPII is analogous to the one for the Korteweg-de Vries equation, as was shown in [B], where local and global well-posedness for KPII in $L^2(\mathbb{R}^2)$ was shown. (For further results, see, for instance, [TT] and the recent work [HHK].) To

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_9,

understand this, consider the associated dispersive functions

$$\omega_{\pm}(\xi,\mu) = \xi^3 - \left(\mp\mu^2/\xi\right),$$

where ω_+ is associated with KPII and ω_- with KPI. These functions appear in the solution operator for the corresponding linear problems. Bourgain's result [B] is obtained (roughly speaking) by using the contraction mapping principle in the space (introduced by Bourgain)

$$X_{\frac{1}{2},+} = \left\{f \in \mathcal{S}'\left(\mathbb{R}^2\right) : \|f\|_{X_{\frac{1}{2},+}} < \infty\right\},$$

where $\|f\|_{X_{\frac{1}{2},+}} = \left(\iint \left|\hat{f}(\xi,\mu,\tau)\right|^2 \left(1+|\tau-\omega_+(\xi,\mu)|\right) d\mu\, d\xi\, d\tau\right)^{1/2}$.

In order to accomplish this, it was fundamental to use the identity

$$\begin{aligned}\Omega_+ &= \omega_+(\xi_1+\xi_2,\mu_1+\mu_2) - \omega_+(\xi_1,\mu_1) - \omega_+(\xi_2,\mu_2)\\ &= \frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left[3(\xi_1+\xi_2)^2 + \left(\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right)^2\right].\end{aligned}$$

Thus, $|\Omega_+| \geq 3|\xi_1|\,|\xi_2|\,|\xi_1+\xi_2|$, which is the same lower bound as for the corresponding quantity for Korteweg–de Vries. (Again see [K2] for further details.) On the other hand, for KPI we have

$$\Omega_- = \frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left[3(\xi_1+\xi_2)^2 - \left(\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right)^2\right]$$

and this has no lower bound in the resonant region $3(\xi_1+\xi_2)^2 = \left(\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right)^2$. Hence, Bourgain's approach in [B] fails for KPI. Similarly, one has

$$|\nabla\omega_+(\xi,\omega)| = \left|\left(3\xi^2+\mu^2/\xi^2, -2\mu/\xi\right)\right| \geq 3|\xi|^2$$

but

$$|\nabla\omega_-(\xi,\mu)| = \left|\left(3\xi^2-\mu^2/\xi^2, 2\mu/xi\right)\right| \geq C|\xi|.$$

Thus, the "local smoothing effect" approach to Korteweg–de Vries, introduced in [KPV] can be used for KPII (see [KZ]), but it fails for KPI. These phenomena were clarified by an example of Molinet–Saut–Tzvetkov (see [MST1, MST2, MST3]). They proved that the flow map for KPI cannot be of class C^2 at the origin, on "any" Sobolev space. Even more, lack of uniform continuity was exhibited in [KT2]. These results imply that local well-posedness cannot be proved for KPI, using the contraction mapping principle, on any Sobolev space. The first subsequent progress in the local and global well-posedness theory of KPI was due to Kenig [K1], who showed that KPI is locally well posed in the space

$$Y_s = \left\{u_0 \in \mathcal{S}'\left(\mathbb{R}^2\right) : \|D_x^s u_0\|_{L^2} < \infty,\ \left\|\partial_y \partial_x^{-1} u_0\right\|_{L^2} < \infty,\ \|u_0\|_{L^2} < \infty\right\},$$

for $s > 3/2$ and globally well posed in the space

$$E_2 = \left\{u_0 : u_0 \in L^2,\ \partial_x^2 u_0 \in L^2,\ \partial_y^2 \partial_x^{-2} u_0 \in L^2,\ \partial_x u_0 \in L^2,\ \partial_y \partial_x^{-1} u_0 \in L^2\right\}.$$

The local well posedness result was obtained by using an enhancement of the energy method, establishing "a priori" control of $\|\partial_x u\|_{L_t^1 L_{x,y}^\infty}$ in terms of $\|u_0\|_{Y_s}$, $s < 3/2$.

This was accomplished by using a frequency-dependent partition of the time interval, first introduced by Vega [V], and applied to the Benjamin–Ono equation in [KT1].

The global well posedness result was obtained by combining the local well posedness result with the conservation law:

$$\begin{aligned}
M(u) &= \int u^2; \\
E(u) &= \frac{1}{2}\int (\partial_x u)^2 + \frac{1}{2}\int \left(\partial_x^{-1}\partial_y u\right)^2 - \frac{1}{6}\int u^3, \\
F(u) &= \frac{3}{2}\int \left(\partial_x^2 u\right)^2 + 5\int (\partial_y u) + \frac{5}{6}\int \left(\partial_x^{-2}\partial_y^2 u\right)^2 - \frac{5}{6}\int u^2 \partial_x^{-2}\partial_y^2 u \\
&\qquad - \frac{5}{6}\int u\left(\partial_x^{-1}\partial_y u\right)^2 + \frac{5}{4}\int u^2 \partial_x^2 u + \frac{5}{24}\int u^4.
\end{aligned}$$

Given the conservation of M and E and the form of the Hamiltonian, the natural long-standing question was the global well posedness in the energy space

$$E = \left\{u_0 : \int (u_0)^2 + \int (\partial_x u_0)^2 + \int \left(\partial_x^{-1}\partial_y u_0\right)^2 < \infty\right\}.$$

We now have the following theorem.

Theorem [IKT] *KPI is globally well posed in E.*

In the remainder of this chapter, I will give a brief description of the proof of the theorem.

In order to implement for KPI the local fixed point argument as in the cases of the Korteweg–de Vries equation or the KPII equation ([KPV], [B]) one would consider the associated linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_y^2 \partial_x^{-1} u = f \\ u|_{t=0} = u_0 \end{cases}. \tag{LKPI}$$

One thinks of $f = \partial_x\left(u^2/2\right)$ and thus, to solve (KPI) on the interval $[0, T]$, $T \leq 1$, one needs to find suitable function spaces $N^1(T)$, $F^1(T)$ for which

i) $\|u\|_{F^1(T)} \leq C\left[\|u_0\|_E + \|f\|_{N^1(T)}\right]$.
ii) $F^1(T) \subset C([0, T]; E)$.
iii) $\|\partial_x\left(u^2/2\right)\|_{N^1(T)} \leq C\|u\|^2_{F^1(T)}$.

It is a simple matter to see that i), ii), iii) give the desired proof by the contraction mapping principle, for u_0 small in E, a situation to which we can easily reduce ourselves, by scaling, due to the subcriticality of the problem at hand. However, as we mentioned before, this is impossible. Our approach here consists of finding spaces $F^1(T)$, $B^1(T)$, and $N^1(T)$, so that if u is a smooth solution of KPI on $\mathbb{R}^2 \times [0, T]$, $T \leq 1$, we have

a) $\|u\|_{F^1(T)} \leq C\left[\|u\|_{B^1(T)} + \|\partial_x\left(u^2/2\right)\|_{N^1(T)}\right]$.
b) $\left\|\partial_x\left(u^2/2\right)\right\|_{N^1(T)} \leq C\|u\|^2_{F^1(T)}$.
c) $\|u\|^2_{B^2(T)} \leq C\left[\|u_0\|^2_E + \|u\|^3_{F^1(T)}\right]$.
d) $F^1(T) \subset C\left([0, T]; E\right)$.

These inequalities and a simple continuity argument suffice to control $\|u\|_{F^1(T)}$, provided $\|u_0\|_E \ll 1$ (which again can always be arranged by scaling). (a) is a linear estimate for solutions of (LKPI), (b) is a "bilinear estimate," and (c) is a frequency localized energy estimate, which exploits the "symmetry" of the non-linear term for real-valued functions. To prove our theorem, we also need to consider differences of solutions and prove analogous estimates for the difference equation. However, for the difference equation, symmetries only hold for L^2 and H^{-1} solutions, so we have to prove some analogous estimates at zero level of regularity. These only hold for functions whose low frequencies are very close to zero, but fortunately, since we are dealing with the difference equation, this can be achieved. Let us conclude with a few words about our new function spaces. The examples in [MST1] show that the bilinear estimate (b) cannot hold in Bourgain spaces, even for solutions of the linear homogeneous problem. To rectify this, we use Bourgain spaces, but only on small, frequency-dependent time intervals. (This is inspired by the recent works [KT], [CCT], and the earlier work [K1] mentioned before.) Thus, if supp $f \subset \left\{|\xi| \sim 2^k\right\}$, we set

$$\|f\|_{X_k^\sigma} = \sum_{j=0}^{\infty} 2^{j/2}\left\|\eta_j\left(\tau - \omega - (\xi, \mu)\right) \cdot \left[(1+|\xi|)^\sigma + \frac{|\mu|}{|\xi|}(1+|\xi|)^{\sigma-1}\right] f(\xi, \mu, \tau)\right\|_{L^2},$$

where η_j is a cut-off function supported on $\left[2^{j-1}, 2^{j+1}\right]$. Also, if supp $\hat{u} \subset \left\{|\xi| \approx 2^k\right\}$, we set

$$\|u\|_{F_k^\sigma} = \sup_{t_k}\left\|\left[u \cdot \eta_0\left(2^k\left(t - t_k\right)\right)\right]^\wedge\right\|_{X_k^\sigma},$$

where η_0 is a cut-off function near 0. The F^σ norm is now defined by

$$\|v\|_{F^\sigma} = \left(\sum_k \left\|\left(\hat{v}\chi_k(\xi)\right)^\vee\right\|^2_{F_k^\sigma}\right)^{1/2},$$

where $\chi_k(\xi)$ is a cut-off to $|\xi| \approx 2^k$. The space N^σ is defined analogously, but introducing also the factor $\left(\tau - \omega_-(\xi,\omega) + i2^{-k}\right)^{-1}$ in the definition of X_k^σ. The spaces $F^\sigma(T)$ and $N^\sigma(T)$ are defined, in the standard way, by restriction. Finally, to define $B^\sigma(T)$ we first define

$$E_\sigma = \left\{ u_0 : \int |\hat{u_0}(\xi,\mu)|^2 \left[1 + |\xi|^\sigma + \frac{|\mu|}{|\xi|} (1+|\xi|)^{\sigma-1} \right]^2 < \infty \right\}.$$

Then

$$\|u\|^2_{B^\sigma(T)} = \|u(0)\|^2_{E_\sigma} + \sum_{k\geq 1} \sup_{t_k \in [0,T]} \left\| u^{\wedge(x,y)}(t_k)\, \chi_k(\xi) \left[(1+|\xi|)^\sigma + \frac{|\mu|}{|\xi|} (1+|\xi|)^{\sigma-1} \right] \right\|^2_{L^2_{\xi,\mu}}.$$

For full details of the proof, see [IKT].

Acknowledgement. This work is supported in part by the NSF.

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On Monge–Ampère Type Equations and Applications

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Summary. This chapter contains an overview of recent results on the Monge–Ampère equation and its linearization. We also describe the reflector problem and regularity results for weak solutions.

Key words: Monge–Ampère equation, weak solutions, regularity, reflector problem.

To Carlos Segovia in memoriam.

1 The Monge–Ampère equation

This equation received attention in recent years and important advances understanding the properties of the solutions and applications to several areas have been found. This chapter describes an overview of these results. The equation is the following:

$$\det D^2u(x) = f(x), \tag{1}$$

where D^2u denotes the Hessian of a function $u : \Omega \to \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ is a domain.

1.1 Basic facts

In general, a fully nonlinear equation $F(D^2u) = f$, where $F : S \to \mathbb{R}$, S being the class of $n \times n$ symmetric matrices, is elliptic on a function $u \in C^2(\Omega)$ if the matrix of second derivatives $F_{r_{ij}}(D^2u(x)) > 0$ for all $x \in \Omega$. When $F(D^2u) = \det D^2u$, we have $F_{r_{ij}}(D^2u(x)) = (D^2u)^{ij} =$ the matrix of cofactors of D^2u, and therefore the Monge–Ampère equation is elliptic in the class of convex functions. In addition, the Monge–Ampère equation is an extremal for all elliptic equations because

C. Cabrelli and J.L. Torrea (eds.), *Recent Developments in Real and Harmonic Analysis*,
Applied and Numerical Harmonic Analysis, DOI 10.1007/978-0-8176-4588-5_10,

$$(\det A)^{1/n} = \inf\{\text{trace}(BA) : B \text{ is symmetric and } \det B = 1\},$$

and is invariant by affine transformations with determinant ± 1 because if A is a matrix and $v(x) = u(Ax)$, then $D^2 v(x) = A^t \left(\left(D^2 u \right) (Ax) \right) A$ and

$$\det D^2 v\ (x) = (\det A)^2\ \det D^2 u(Ax).$$

It gives us a great advantage to understand (1) in a weak sense as follows. The *normal or gradient map, or subdifferential* of a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, is the multi-valued map defined by

$$\partial u(y) = \{p \in \mathbb{R}^n : u(x) \geq u(y) + p \cdot (x - y), \forall x \in \Omega\},$$

and $\partial u(E) = \cup_{y \in E} \partial u(y)$ for each $E \subset \Omega$. We have the following.

- $\partial u(E)$ is compact for each $E \subset \Omega$ compact;
- The *Legendre transform* is defined by

$$u^*(y) = \sup_{x \in \Omega} x \cdot y - u(x).$$

  u^* always convex so differentiable a.e.;
- If $u \in C(\bar{\Omega})$, then the set

$$F = \{p \in \mathbb{R}^n : p \in \partial u(x) \cap \partial u(y), x \neq y\}$$

  has Lebesgue measure zero. This follows because F is contained in the set of points where u^* is not differentiable;
- This yields that ∂u is single valued a.e.;
- $\partial(\Omega \setminus E) = (\partial(\Omega) \setminus \partial u(E)) \cup (\partial u(\Omega \setminus E) \cap \partial u(E))$.

Combining these items we obtain that the class $\mathcal{S} = \{E \subset \Omega : \partial u(E)$ is Lebesgue measurable$\}$ is a Borel σ-algebra, and

$$Mu(E) = |\partial u(E)|$$

is σ-additive over $\mathcal{S}$; Mu is called the *Monge–Ampère measure* associated with u. This generalizes (1) because if $u \in C^2(\Omega)$ and convex, then $Mu(E) = \int_E \det D^2 u(x)\, dx$. The notion of a weak solution to (1) is then defined as follows. Let ν be a Borel measure on Ω, the convex function u is a *weak solution or Aleksandrov* solution to (1) if

$$Mu = \nu.$$

The following two theorems are extremely important in all the theory, and they are based on the following basic property [Gut01, Lemma 1.4.1]:

$$\text{if } v = u \text{ on } \partial\Omega, \text{ and } v \geq u \text{ in } \Omega \Longrightarrow \partial v(\Omega) \subset \partial u(\Omega).$$

Theorem 1 *(Aleksandrov maximum principle) [Gut01, Theorem 1.4.2]*
If Ω is open convex bounded, $u \in C(\bar{\Omega})$ convex, and $u = 0$ on $\partial\Omega$, then

$$|u(y)|^n \leq C_n \,\mathrm{diam}(\Omega)^{n-1}\, dist(y, \partial\Omega)\, Mu(\Omega), \quad \forall y \in \Omega.$$

Proof If v is the cone with vertex $(y, u(y))$ and base $\Omega \times \{0\}$, then $v \geq u$ in Ω, and $u = v$ on $\partial\Omega$. Next we have *convex hull* $\left\{B\left(0, \frac{-u(y)}{\mathrm{diam}(\Omega)}\right), p\right\} \subset \partial v(\Omega) \subset \partial u(\Omega)$, with $|p| = -\frac{u(y)}{\mathrm{dist}(y, \partial\Omega)}$. Calculating the volumes, the theorem follows. □

Theorem 2 *(Comparison principle) [Gut01, Theorem 1.4.6] If $u, v \in C(\bar{\Omega})$ with $Mu \leq Mv$, then*

$$\min_{\bar{\Omega}}(u - v) = \min_{\partial\Omega}(u - v).$$

1.2 The Dirichlet problem

The Dirichlet problem is solvable for the Monge–Ampère equation. More precisely, we have the following results.

Theorem 3 *[Gut01, Theorem 1.5.2] Let Ω be strictly convex, $g \in C(\partial\Omega)$. Then there exists a unique $u \in C(\overline{\Omega})$ convex weak solution to*

$$\begin{aligned} Mu &= 0 \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

Proof Let

$$\mathcal{F} = \{a(x) \text{ affine with } a \leq g \text{ on } \partial\Omega\},$$

the solution is

$$u(x) = \sup\{a(x) : a \in \mathcal{F}\}.$$

Indeed, since Ω is strictly convex, $u = g$ on $\partial\Omega$; and the linear nature of u implies that $\partial u(\Omega) \subset \{p \in \mathbb{R}^n : p \in \partial u(x) \cap \partial u(y), x \neq y\}$, which has measure zero. □

Theorem 4 *[Gut01, Theorem 1.6.2] Let Ω be strictly convex, μ be a Borel measure over Ω with $\mu(\Omega) < \infty$, and $g \in C(\partial\Omega)$. Then there exists a unique $u \in C(\overline{\Omega})$ convex weak solution to*

$$\begin{aligned} Mu &= \mu \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

Proof (Outline of the proof) It is based in the Perron method. Let

$$\mathcal{F}(\mu, g) = \{v \in C(\bar{\Omega}) : v \text{ convex}, Mv \geq \mu, v = g \text{ on } \partial\Omega\},$$

and let W solve $MW = 0$, $W = g$ on $\partial\Omega$. Then $MW \leq \mu \leq Mv$ for $v \in \mathcal{F}(\mu, g)$, and so by the comparison principle,

$$v \leq W.$$

If $\mathcal{F}(\mu, g) \neq \emptyset$, then let

$$U(x) = \sup\{v(x) : v \in \mathcal{F}(\mu, g)\}.$$

This is the solution. To prove this we proceed by approximation: approximate μ by a sequence of measures of the form $\mu_N = \sum_{i=1}^N a_i\, \delta_{x_i}$ with $a_i > 0$ and $x_i \in \Omega$. We have the following.

1. $\mathcal{F}(\mu_N, g) \neq \emptyset$. Notice that $M(|x - x_i|) = \omega_n\, \delta_{x_i}$. Let $f(x) = \frac{1}{\omega_n^{1/n}} \sum_{i=1}^N a_i^{1/n}$ $|x - x_i|$. Solve $Mu = 0$ in Ω with $u = g - f$ on $\partial\Omega$. The function $v = u + f$ belongs to $\mathcal{F}(\mu_N, g)$, because $Mv \geq Mu + Mf$.
2. $u, v \in \mathcal{F}(\mu_N, g) \Longrightarrow u \vee v \in \mathcal{F}(\mu_N, g)$.
3. $U \in \mathcal{F}(\mu_N, g)$.
4. $MU \leq \mu$. To do this define the "lifting" of U, solve $Mv = 0$ in $B_r(x_0)$ with $v = U$ on $\partial B_r(x_0)$, and let

$$w(x) = \begin{cases} v(x) \text{ for } x \in B_r(x_0) \\ U(x) \text{ for } x \in B_r(x_0)^C \end{cases}.$$

□

1.3 Regularity of solutions

If $L = D_i(a_{ij}(x)D_j)$ or $L = a_{ij}(x)D_{ij}$ satisfies the uniform ellipticity condition

$$\lambda\, Id \leq (a_{ij}(x)) \leq \Lambda\, Id,$$

then a regularity theory for solutions of $Lu = f$ is available; in the first case the De Giorgi–Nash–Moser theory [Gio57], [Nas58], [Mos61], and in the second case the Krylov–Safonov theory [KS81]. For the Monge–Ampère equation, the ellipticity in linear equations is replaced by

$$\lambda\, |E| \leq Mu(E) \leq \Lambda\, |E|.$$

This implies a regularity theory for solutions of $Mu = f$. For example, assuming appropriate boundary data, Aleksandrov solutions are $C^{1,\alpha}$. The regularity theory is based on the notion of section and normalization. The *sections* of u are the convex sets

$$S(y,t) = \{x \in \Omega : u(x) < u(y) + Du(y) \cdot (x-y) + t\}.$$

The convex domain Ω is normalized if $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$. By a result of Fritz John, given Ω open, bounded, and convex, there exists T affine such that $T(\Omega)$ is normalized. The sections satisfy geometric properties that yield estimates of solutions using measure theoretical arguments. Mainly covering lemmas which yield distribution function estimates (harmonic analysis techniques), see [Gut01, Chapter 3]. For example, the sections have the following properties that show that they behave like Euclidean balls after affine transformations. Assume $\lambda \le \det D^2u \le \Lambda$, $u = 0$ on $\partial\Omega$, then

- Let T be affine normalizing $S_u(x,t)$, $T\left(S_u(x,t)\right) = S_v(Tx,t)$ where $v(y) = u(T^{-1}y)$.
- Engulfing property: There exists a constant $\vartheta > 0$ such that

$$y \in S(z,t) \Longrightarrow S(z,t) \subset S(y, \vartheta t).$$

- Let

$$\Omega_\alpha = \{x \in \Omega : u(x) < (1-\alpha) \min_{\Omega} u\}.$$

  Given $0 < \alpha < 1$, there exists $\eta(\alpha) > 0$ such that if $x \in \Omega_\alpha$ and $t \le \eta(\alpha)$, then $S(x,t) \Subset \Omega$.
- There exist positive constants K_1, K_2, K_3 and ϵ such that if $S(z_0, r_0)$ and $S(z_1, r_1)$ are sections with $r_1 \le r_0$, $S(z_0, r_0) \cap S(z_1, r_1) \ne \emptyset$, and T is an affine transformation that normalizes $S(z_0, r_0)$, then

$$B\left(Tz_1, K_2 \frac{r_1}{r_0}\right) \subset T(S(z_1, r_1)) \subset B\left(Tz_1, K_1 \left(\frac{r_1}{r_0}\right)^{\epsilon}\right),$$

  and $Tz_1 \in B(0, K_3)$.
- Sections are strictly convex.

We mention the following regularity results, assuming that Ω is a strictly convex normalized domain, and $\det D^2u(x) = f(x)$, $u = 0$ on $\partial\Omega$. We have the following.

- $f = 1$; then L^∞-estimates hold (Pogorelov, [Pog71]):

$$C_1\, Id \le D^2u(x) \le C_2\, Id, \quad x \in \Omega'$$

  where $\Omega' \Subset \Omega$, and C_i are positive constants depending only on the domains.
- f continuous, $\lambda \le f \le \Lambda$; then Caffarelli proved L^p interior estimates, see [Caf90] and [Gut01, Chapter 6]:

$$\|D^2u\|_{L^p(\Omega')} \le C,$$

  where $\Omega' \Subset \Omega$, and C is a positive constant depending on the domains. In addition, if $f \in C^\alpha$ then $u \in C^{2,\alpha}$.
- f has a modulus of continuity w_f satisfying a Dini condition; then $D^2u \in L^\infty(\Omega')$, and $\exists\, f \in C(\Omega)$, with $D^2u \notin L^\infty$ (X.-J. Wang, [Wan95]).

1.4 Estimates for the linearized Monge–Ampère equation

Let $u : \mathbb{R}^n \to \mathbb{R}$. We have

$$\det D^2 (u + tv) =$$

$$\det D^2 u + t \operatorname{trace}\left(U D^2 v\right) + \cdots + t^n \det D^2 v,$$

where $U = (U)_{ij}$ is the matrix of cofactors of $D^2 u$. The *linearized Monge–Ampère* operator is

$$L_U v = \operatorname{trace}\left(U D^2 v\right).$$

We have $L_U u = n \det D^2 u, \operatorname{div}\,(UDv) = \operatorname{trace}(UD^2 v)$, and $D^2 u$ is positive semi-definite; then U is positive semi-definite; Thus, L_U is elliptic, possibly degenerate. L_U appears, for example, when one considers first derivatives of u: $u_j = D_j u$, $j = 1, \ldots, n$, and we have the equation

$$L_U(u_j) = \mu_j.$$

Theorem 5 *(L. A. Caffarelli and C. E. Gutiérrez) Suppose*

$$\lambda \leq \det D^2 u \leq \Lambda.$$

There exists a constant C depending only on n, λ, and Λ such that

$$\sup_S v \leq C \inf_S v,$$

for all sections S of u and for each solution $v \geq 0$ to $L_U v = 0$.

The proof of this theorem uses the geometric properties of the sections already mentioned, see [CG96], [CG97]. As a consequence, of this theorem, we obtain Hölder estimates. These estimates are useful in affine geometry, see [TW00].

To put the results of the next part in perspective, we mention the following theorem.

Theorem 6 *(Caffarelli, [Caf91]) Suppose u convex in Ω is a weak solution to the Monge–Ampère equation*

$$\lambda \leq \det D^2 u \leq \Lambda.$$

If ℓ is a supporting hyperplane to u, and the set $\Gamma = \{x : u(x) = \ell(x)\}$ has more than one point, then Γ has no extremal points in the interior of Ω.

This implies that solutions to the Monge–Ampère with $u = 0$ on $\partial\Omega$ are strictly convex.

2 A Monge–Ampère type equation for reflectors

2.1 Snell's law

Snell's law of reflection on a perfectly reflecting plane says that

$$\text{angle(normal, incident ray)} = \text{angle(normal, reflected ray)}.$$

If we have a perfectly reflecting surface S, then Snell's law of reflection reads:

$$\begin{aligned}&\text{angle(normal to tangent plane, incident ray)}\\&= \text{angle(normal to tangent plane, reflected ray)}.\end{aligned}$$

For example, the reflection on some simple surfaces is as follows:

- Sphere: rays emanating from the center bounce back to the center.
- Ellipsoid: rays emanating from one focus go to the other focus.
- Paraboloid: rays emanating from the focus are reflected into parallel rays.

There is a formula for the reflected rays as follows. If $\Omega \subset S^{n-1}$ and a ray emanates from O in the direction $x \in \Omega$, and $\mathcal{A}$ is a reflecting surface parameterized by

$$z = x\,\rho(x),$$

then the reflected ray has direction:

$$T(x) = x - 2\,\langle x, \nu\rangle\,\nu,$$

where ν is the outer normal to $\mathcal{A}$ at $z = x\,\rho(x)$.

2.2 The reflector problem

The *reflector problem* is the following. Let O be a light source; Ω, Ω^* are two domains in the sphere S^{n-1}; $f(x)$ is a positive function for $x \in \Omega$ (input illumination intensity); $g(x^*)$ is a positive function for $x^* \in \Omega^*$ (output illumination intensity). Suppose that light emanates from O with intensity $f(x)$ for $x \in \Omega$. Find a reflecting surface $\mathcal{A}$ parameterized by $z = x\,\rho(x)$ for $x \in \Omega$, such that all reflected rays by $\mathcal{A}$ fall in the direction Ω^*, and the output illumination received in the direction x^* is $g(x^*)$. Assuming no loss of energy in the reflection, we have by conservation of energy that

$$\int_\Omega f(x)\,dx = \int_{\Omega^*} g(x)\,dx.$$

In addition, and again by conservation of energy, T is a measure-preserving map:

$$\int_{T^{-1}(E)} f(x)\,dx = \int_E g(x)\,dx, \qquad \text{for all Borel sets } E \subset \Omega^*.$$

The reflector problem can then be formulated as follows. Given Ω, Ω^* domains on S^{n-1}, $f \in C(\Omega)$, $g \in C(\Omega^*)$ positive functions, find a reflecting surface $\mathcal{A}$ parameterized by $z = x\,\rho(x)$ for $x \in \Omega$ such that

$$T(\Omega) = \Omega^*.$$

Using the fact that T is a measure-preserving map, the Jacobian of T is $\dfrac{f(x)}{g(T(x))}$. Taking an orthonormal frame of coordinates on S^{n-1} yields the partial differential equation (PDE)

$$\mathcal{L}\rho = \eta^{-2} \det\left(-\nabla_i\nabla_j\rho + 2\,\rho^{-1}\nabla_i\rho\nabla_j\rho + (\rho - \eta)\delta_{ij}\right) = \frac{f(x)}{g(T(x))};$$

where ∇ = covariant derivative, $\eta = \dfrac{|\nabla\rho|^2 + \rho^2}{2\rho}$, and δ_{ij} is the Kronecker delta.

This is a very complicated, fully nonlinear PDE of Monge–Ampère type, like the Monge–Ampère equation, it is easier to work with weak solutions. The reflector equation has received attention from engineering and numerical points of view and is useful in several applications. A weak formulation of a solution was introduced by Xu-Jia Wang [Wan96] and also by L. Caffarelli and V. Oliker [CO94], who proved existence. In the smooth case, Pengfei Guan and Xu-Jia Wang [GW98] proved a priori estimates and the existence of smooth solutions using the method of continuity. A connection with mass transportation in the sphere was found by Xu-Jia Wang [Wan04], with the cost function $c(x, y) = -\log(1 - x \cdot y)$. A connection with mass transportation was also found in the case of two reflectors by T. Glimm and V. Oliker [OG03].

2.3 Notion of weak solution for the reflector problem

Let $m \in S^{n-1}$, and $b > 0$. $P(m, b)$ denotes the paraboloid of revolution in $\mathbb{R}^n$ with focus at 0, axis m, and directrix plane $m \cdot x + 2b = 0$.

Given $\rho(x) \in C(S^{n-1})$, $\rho > 0$, the radial reflector associated with ρ is

$$\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}.$$

$P(m, b)$ is a supporting paraboloid for the reflector antenna $\mathcal{A}$ at the point $z_0 = x_0\rho(x_0)$ if $z_0 \in P(m, b)$ and $\mathcal{A}$ is contained in the interior region limited by the surface described by $P(m, b)$. The antenna $\mathcal{A}$ is an admissible surface if it has a supporting paraboloid at each point. We can then define a notion similar to the normal mapping. Given an admissible antenna $\mathcal{A}$ and $x_0 \in S^{n-1}$, the *reflector mapping* associated with $\mathcal{A}$ is

$$\mathcal{N}_{\mathcal{A}}(x_0) = \{m \in S^{n-1} : P(m, b) \text{ is a support of } \mathcal{A} \text{ at } x_0\rho(x_0)\}.$$

If $E \subset S^{n-1}$, then $\mathcal{N}_{\mathcal{A}}(E) = \cup_{x_0 \in E}\mathcal{N}_{\mathcal{A}}(x_0)$. The class of sets $E \subset S^{n-1}$ for which $\mathcal{N}_{\mathcal{A}}(E)$ is Lebesgue measurable is a Borel σ-algebra.

Given $g \in L^1(S^{n-1})$ we define the Borel measure

$$\mu_{g,\mathcal{A}}(E) = \int_{\mathcal{N}_{\mathcal{A}}(E)} g(x)\, dx.$$

The surface $\mathcal{A}$ is a *weak solution of the reflector antenna equation* if

$$\mu_{g,\mathcal{A}}(E) = \int_E f(x)\, dx,$$

for each Borel set $E \subset S^{n-1}$. Smooth solutions to $\mathcal{L}\rho = f/g$ are weak solutions.

A natural question is then, how regular are weak solutions?

2.4 Results

Theorem 7 *([CGH08]) Suppose $\mathcal{A}$ is a reflector antenna given by $\mathcal{A} = \{x\rho(x) : x \in S^{n-1}\}$, and satisfying the following.*

1. *there exist constants r_1, r_2 such that $0 < r_1 \le \rho(x) \le r_2$ for all $x \in S^{n-1}$; and*
2. *there exist positive constants C_1, C_2 such that*

$$C_1 \le \frac{|\mathcal{N}_{\mathcal{A}}(E)|}{|E|} \le C_2, \text{ for each Borel set } E \subset S^{n-1}.$$

If $P(m,a)$ is a supporting paraboloid to $\mathcal{A}$, then $P(m,a) \cap \mathcal{A}$ is a single point set. In addition, $\mathcal{A}$ is a C^1 surface.

Proof (Idea of the proof)

Let $m \in S^{n-1}$, $b > 0$, and let $P(m,b)$ be the paraboloid in $\mathbb{R}^n$ with focus at 0, axis m, and directrix hyperplane $\Pi(m,b)$ of equation $m \cdot y + 2b = 0$. $P(m,b)$ has the equation

$$P(m,b) \equiv \{y : |y| = m \cdot y + 2b\}.$$

An *important simple fact* is the following: if $P(m',b')$ is another paraboloid, then $P(m,b) \cap P(m',b')$ is contained in the bisector of the directrices of both paraboloids, denoted by $\Pi[(m,b),\,(m',b')]$ and whose equation is

$$\Pi[(m,b),\,(m',b')] \equiv \{y : (m - m') \cdot y + 2(b - b') = 0\}.$$

For the proof we proceed to show an upper and lower estimate of the reflector mapping. First we discuss the upper estimate. Consider the portion $\mathcal{R}$ of the antenna $\mathcal{A}$ between the paraboloids $P(e_n, a)$ and $P(e_n, a+h)$ with $h > 0$ which is a supporting paraboloid to $\mathcal{A}$, and where e_n is the nth coordinate vector in $\mathbb{R}^n$. Suppose that $P(m,b)$ is a supporting paraboloid to $\mathcal{A}$ at some point y_0 in the portion $\mathcal{R}$. The goal is to estimate the size of the axis m. This is precisely described in the following proposition.

Proposition 1 *Let E be the ellipsoid of minimum volume containing the projection over $\mathbb{R}^{n-1}$ of $\mathcal{R}$. Suppose that E has principal axes $\lambda_1, \cdots, \lambda_{n-1}$ in the coordinate directions $e_1, \dots, e_{n-1}$. Assume $d_1 \le \operatorname{diam}(E) \le d_2$.*

If $\mathcal{R}_{1/2}$ is the portion of $\mathcal{R}$ whose projection on $\mathbb{R}^{n-1}$ is the contracted ellipsoid $(1/2(n-1))E$, then

$$\mathcal{N}_{\mathcal{A}}(\mathcal{R}_{1/2}) \subset$$

$$\{m \in S^{n-1}; |m_i| \le C\, h/\lambda_i, \quad i = 1, \dots, n-1, \\ \text{and } |m'| \le \sqrt{2(1-m_n)} \le C\,\sqrt{h}/\operatorname{diam}(E)\}.$$

Next we consider the lower estimate, which is similar to the Aleksandrov estimate for the normal mapping, Theorem 1.

Proposition 2 *E is the ellipsoid of minimum volume containing the projection over $\mathbb{R}^{n-1}$ of $\mathcal{R}$. Suppose that E has principal axes $\lambda_1, \dots, \lambda_{n-1}$ in the coordinate directions $e_1, \dots, e_{n-1}$, and $d_1 \le \operatorname{diam}(E) \le d_2$. Let $z' = (z_1, \dots, z_{n-1}) \in \mathcal{R}'$ such that $z = (z', z_n) \in \mathcal{R} \cap P(e_n, a+h)$ with $K - \delta_z \lambda_1 \le z_1 \le K$, where $K = \sup_{x' \in \mathcal{R}'} x_1$, i.e., z' is close to $\partial \mathcal{R}'$.*

Then there exists ϵ_0 such that

$$\{m \in S^{n-1} : \sqrt{1-m_n} \le \epsilon_0 \frac{\sqrt{h}}{\operatorname{diam}(E)}, \\ 0 \le -m_1 \le \epsilon_0 \frac{h}{\delta_z\, \lambda_1}, \; |m_i| \le \epsilon_0 \frac{h}{\lambda_i} \text{ for } i = 2, \dots, n-1\} \\ \subset \mathcal{N}_{\mathcal{A}}(\mathcal{R}).$$

With these two propositions, we proceed to prove the first part of the theorem.

$$\Delta = \text{ projection of } P(e_n, a_1) \cap \mathcal{A} \text{ on } \mathbb{R}^{n-1}$$

If Δ contains at least two points, then $\operatorname{diam}(\Delta) = C$.

$$\begin{aligned} \mathcal{R}_h &= \text{ portion of } \mathcal{A} \text{ cut by } P(e_n, a_1 - h) \\ \mathcal{R}'_h &= \text{ projection of } \mathcal{R}_h \text{ on } \mathbb{R}^{n-1} \\ E_h &= \text{ John's ellipsoid for } \mathcal{R}'_h \\ (\mathcal{R}_h)_{1/2} &= \text{ lower portion of } \mathcal{R}_h \text{ over } \frac{1}{2(n-1)} E_h \\ D_h &= \text{ preimage of } \mathcal{R}_h \text{ in } S^{n-1} \\ (D_h)_{1/2} &= \text{ preimage of } (\mathcal{R}_h)_{1/2} \text{ in } S^{n-1} \end{aligned}$$

We have that $\mathcal{R}_h \to \mathcal{R}_0$ in the Hausdorff metric and so $\mathcal{R}'_h \to \Delta$ in the same metric.

Let $\lambda_1(h)$ be the longest axis of E_h, then $\lambda_1(h) \approx \operatorname{diam}(\Delta)$. There exists $z_h \in \Delta$ such that $K - \delta_h \lambda_1(h) \leq (z_h)_1 \leq K = \sup_{z \in \mathcal{R}'_h} z_1$, and $\delta_h \to 0$.

We have

$$|D_h| \approx |\mathcal{R}'_h|$$
$$|(D_h)_{1/2}| \approx |(\mathcal{R}_h)_{1/2}| \approx |E_h|.$$

Then from Proposition 1,

$$C\,|E_h| \leq |\mathcal{N}((\mathcal{R}_h)_{1/2})| \leq \min\left\{\frac{C\,h}{\lambda_1}, \frac{C\,\sqrt{h}}{\operatorname{diam}(E_h)}\right\}$$
$$\prod_{i=2}^{n-1} \min\left\{\frac{C\,h}{\lambda_i}, \frac{C\,\sqrt{h}}{\operatorname{diam}(E_h)}\right\},$$

and from Proposition 2,

$$|E_h| \geq |\mathcal{R}'_h| \geq C\,|\mathcal{N}_{\mathcal{A}}(D_h)| \geq C \min\left\{\frac{\epsilon_0\,h}{\delta_h\,\lambda_1}, \frac{\epsilon_0\,\sqrt{h}}{\operatorname{diam}(E_h)}\right\}$$
$$\prod_{i=2}^{n-1} \min\left\{\frac{\epsilon_0\,h}{\lambda_i}, \frac{\epsilon_0\,\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

Therefore,

$$\epsilon_0^{n-1} \min\left\{\frac{h}{\delta_h\,\lambda_1}, \frac{\sqrt{h}}{\operatorname{diam}(E_h)}\right\} \leq C \min\left\{\frac{h}{\lambda_1}, \frac{\sqrt{h}}{\operatorname{diam}(E_h)}\right\}.$$

Since $\lambda_1 \approx \operatorname{diam}(E_h) \approx \mathit{const}$, we obtain the following contradiction:

$$\epsilon_0^{n-1} \min\left\{\frac{h}{\delta_h}, \sqrt{h}\right\} \leq C\,h,$$

for any $h > 0$ sufficiently small.

□

Acknowledgements

These are notes from the mini-course I gave in the conference in honor of Carlos Segovia in December 2005 in Buenos Aires. It is a pleasure to thank all the colleagues and friends for the opportunity to give these talks and for putting together this wonderful event in honor of our good friend Carlos Segovia. It is extremely sad that he passed away shortly after that in April of 2007.

The author was partially supported by NSF grant DMS-0300004.

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